Solving Dynamic General Equilibrium models by hand

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There are few instances in which a dynamic general equilibrium macro model will have a closed form solution. As exemplified by Brock and Mirman (1972) and Levhari and Srinivasan (1969), these models rely on logarithmic utility and full depreciation of capital or, alternatively, linear constraints and quadratic preferences. This note will cover such cases and provide examples of two solution methods: value function iteration and guess and verify when a closed form solution is available.

1 Deterministic Model

Consider this simple, centralized and deterministic version of a neoclassical growth model. A planner chooses sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility:

$$\max U = \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to:

$$k_t^{\alpha} \geq c_t + k_{t+1}$$
$$c_t > 0 \;\; \forall t \geq 0$$

with $\alpha \in (0,1), \beta \in (0,1)$ and $k_0 > 0$ given.

Note that both the resource constraints and the utility function are assumed to be concave, so the planner will solve a concave optimization problem.

1.1 Guess and Verify

This method relies on the fact that an optimization problem's value function is unique. The idea is to guess a particular functional form of a solution and then verify that the solution has in fact this form. This method works particularly well for simpler models like the one at hand.

The Bellman equation for the Planner's problem is:

$$\begin{aligned} V(k) &= \max_{(c,k')} \{\ln c + \beta V(k')\} \\ s.t. &: \;\; k^{\alpha} \geq c + k' \end{aligned}$$

*DISCLAIMER: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!

$$\Rightarrow V(k) \quad = \quad \max_{k'}\{\ln(k^{\alpha} - k') + \beta V(k')\} \tag{1}$$

Let us guess that the value function has the following shape:

$$V(k) = A + B\ln k \tag{2}$$

where A and B are coefficients to be determined.

If we substitute our guess into (1), the right hand side (RHS) of the equation is then:

$$RHS = \max_{k'}\{\ln(k^{\alpha} - k') + \beta[A + B\ln k']\} \tag{3}$$

Taking first order conditions yields:

$$\frac{\beta B}{k'} \quad = \quad \frac{1}{(k^{\alpha} - k')}$$

$$\Rightarrow \quad k' = \frac{\beta B k^{\alpha}}{1 + \beta B} \tag{4}$$

Replace the above to re-write (3) in terms of the optimal value of k':

$$V(k) = \left\{\ln\left(k^{\alpha} - \frac{\beta B k^{\alpha}}{1 + \beta B}\right) + \beta\left[A + B\ln\frac{\beta B k^{\alpha}}{1 + \beta B}\right]\right\} \tag{5}$$

Given the above and our guess $V(k) = A + B\ln k$ we can write:

$$A + B\ln k = \left\{\underbrace{\ln\left(k^{\alpha} - \frac{\beta B k^{\alpha}}{1 + \beta B}\right)}_{\text{Term I}} + \beta A + \underbrace{\beta B\ln\left(\frac{\beta B k^{\alpha}}{1 + \beta B}\right)}_{\text{Term II}}\right\} \tag{6}$$

Term I:

$$\ln\left(k^{\alpha} - \frac{\beta B k^{\alpha}}{1 + \beta B}\right) = \ln\left(\frac{k^{\alpha}}{1 + \beta B}\right) = \alpha\ln k - \ln(1 + \beta B)$$

Term II:

$$\begin{aligned}\beta B\ln\left(\frac{\beta B k^{\alpha}}{1 + \beta B}\right) &= \beta B\ln k^{\alpha} + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right) \\ &= \beta B\alpha\ln k + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right)\end{aligned}$$

Hence (6) can be written as

$$A + B\ln k = \alpha\ln k - \ln(1 + \beta B) + \beta A + \beta B\alpha\ln k + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right)$$

Collecting all terms with k, we can rearrange the RHS as follows:

$$A + B\ln(k) = [\alpha + \beta B\alpha]\ln(k) - \ln(1+\beta B) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) \tag{7}$$

It is now possible to match the coefficients in this expression with the coefficients in the guess-and-verify expression. The implication is that:

$$\begin{aligned} B\ln(k) &= [\alpha + \beta B\alpha]\ln(k) \\ &\Rightarrow \; B = [\alpha + \alpha\beta B] \\ &\Rightarrow \; B = \frac{\alpha}{1-\alpha\beta} \end{aligned}$$

Plugging the above back into (4) yields the policy function for capital:

$$\begin{aligned} k' &= \left[\frac{\beta B}{1+\beta B}\right] k^{\alpha} \\ &= \left[\frac{\beta\left(\frac{\alpha}{1-\beta\alpha}\right)}{1+\beta\left(\frac{\alpha}{1-\beta\alpha}\right)}\right] k^{\alpha} \\ &= \alpha\beta k^{\alpha} \end{aligned}$$

Replacing the above into the resource constraint $c = k^{\alpha} - k'$, yields the agent's policy function for consumption:

$$c = (1-\alpha\beta)k^{\alpha}$$

Last, plugging the value of B into (7) yields the value of A. This is a verifying step to make sure that both terms A and B are indeed constants as assumed when guessing the functional form.

1.2 Value Function Iteration

In the last section we began with a guess, parametrized it and used the method of undetermined coefficients (guess-and-verify) to solve for the solution of the functional equation. Now, we will try an iterative procedure which is based on the Contraction Theorem. Since we already know the solution to this example from the previous section, we can use this opportunity as a way to double check our answers.

Consider the planner's Bellman equation at $t = 1$:

$$V_1(k) = \max_{k'}\{\ln(k^{\alpha} - k') + \beta V_0(k')\}$$

First guess an arbitrary function $V_0(k)$. For concreteness, assume $V_0(k) = 0 \;\; \forall k$. The above then becomes:

$$V_1(k) = \max_{k'}\{\ln(k^{\alpha} - k')\}$$

Proceed recursively by solving for the optimal value of k'. Clearly, the function above is maximized by choosing $k' = 0$. This implies that

$$\begin{aligned} V_1(k) &= \ln(k^\alpha - 0) + \beta V_0(0) \\ &= \ln k^\alpha \\ &\Rightarrow V_1(k) = \alpha \ln k \end{aligned}$$

Given that we now know V_1 we can proceed to solve:

$$v_2(k) = \max_{0 \le k' \le k^\alpha} (\ln(k^\alpha - k') + \beta v_1(k'))$$

$$v_2(k) = \max_{0 \le k' \le k^\alpha} \{\ln(k^\alpha - k') + \beta\alpha \ln k'\}$$

Taking FOC with respect to capital:

$$[k'] : \frac{1}{k^\alpha - k'} = \frac{\alpha\beta}{k'}$$

$$\Rightarrow k' = \frac{\alpha\beta k^\alpha}{1 + \alpha\beta}$$

$$\Rightarrow c = \left(\frac{1}{1 + \alpha\beta}\right) k^\alpha$$

substituting the policy function for k' yields:

$$\begin{aligned} V_2(k) &= \left\{\ln\left(k^\alpha - \frac{\alpha\beta k^\alpha}{1 + \alpha\beta}\right) + \beta\alpha \ln\left(\frac{\alpha\beta k^\alpha}{1 + \alpha\beta}\right)\right\} \\ &= \ln\left(\frac{1}{1 + \alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1 + \alpha\beta}\right) + \alpha(1 + \alpha\beta) \ln k \end{aligned}$$

In turn, by iterating on the recursion:

$$V_{n+1}(k) = \max_{0 \le k' \le k^\alpha} (\ln(k^\alpha - k') + \beta V_n(k'))$$

we can obtain a sequence of value functions $\{v_n\}_{n=0}^\infty$ and policy functions $\{g_n\}_{n=1}^\infty$ which will converge to the solution v^* and the associated policy function g^* (see Stockey and Lucas Chapter 3 for the conditions for the existence of a contraction as well as the proof of the contraction mapping theorem).

Using the algebra of geometric series we can obtain the limiting policy and value functions:

$$\begin{aligned} c &= (1 - \alpha\beta) k^\alpha \\ k' &= \alpha\beta k^\alpha \\ V(k) &= \frac{1}{1 - \beta}\left[\ln(1 - \alpha\beta) + \left(\frac{\alpha\beta}{1 - \alpha\beta}\right) \ln(\alpha\beta)\right] + \frac{\alpha}{1 - \alpha\beta} \ln k \end{aligned}$$

Note the above equation shows that the optimal policy is to have capital move according to the difference equation $k_{t+1} = \alpha\beta k_t^\alpha$ or $\ln k_{t+1} = \ln \alpha\beta + \alpha \ln k_t$. Given that $\alpha < 1$, we have assumed that k_t

will converge as $t \to \infty$ for any initial value of k_0.

Last, note that both policy functions are identical to the ones obtained in the previous section confirming our initial results. Moreover, constants $B = \frac{\alpha}{1-\alpha\beta}$ and $A = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)\right]$ are also equivalent.

2 Stochastic Model

Consider an alternative version of the neoclassical growth model, now decentralized and with households and firms that face uncertainty.

The Household problem

$$\begin{aligned} \max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t [\ln c_t + \gamma \ln(1-h_t)] \\ s.t. &: \quad r_t k_t + w_t h_t \geq c_t + k_{t+1} \end{aligned}$$

where $\gamma > 0$ is a measure of the utility of leisure (or desutility of labor).

The Firm problem

$$\begin{aligned} \max_{(k_t, h_t)} \pi_t &= p_t y_t - w_t h_t - r_t k_t \\ s.t. &: \quad y_t = z_t k_t^{\alpha} h_t^{1-\alpha} \end{aligned}$$

where z_t is a stochastic productivity shock.

2.1 Guess and Verify

The household's bellman equation would be given by:

$$V(k_t, z_t) = \max_{(k_{t+1}, h_t)} \{\ln(r_t k_t + w_t h_t - k_{t+1}) + \gamma \ln(1-h_t) + \beta E_t V(k_{t+1}, z_{t+1})\}$$

or using the standard "prime" notation to specify one period ahead variables:

$$V(k, z) = \max_{k', h} \{\ln(rk + wh - k') + \gamma \ln(1-h) + \beta EV(k', z')\}$$

Since we have two state variables now, our functional guess must be different to the one in the previous section. In particular, assume our guess is:

$$V(k, z) = A + B \ln k + C \ln z$$

Rewrite the Bellman equation as:

$$V(k_t, z_t) = \max_{k_{t+1}, h_t} \{\ln(r_t k_t + w_t h_t - k_{t+1}) + \gamma \ln(1 - h_t) + \beta E_t[A + B\ln(k_{t+1}) + C\ln(z_{t+1})]\}$$

Take first order conditions:

$$\frac{\delta V}{\delta k_{t+1}} = 0 \iff \frac{1}{\underbrace{r_t k_t + w_t h_t}_{y_t} - k_{t+1}} = \beta B \frac{1}{k_{t+1}}$$

$$\Rightarrow \quad k_{t+1} = y_t \underbrace{\left(\frac{\beta B}{1 + \beta B}\right)}_{\text{Saving Rate } <1}$$

$$\frac{\delta V}{\delta h_t} = 0 \iff \left(\frac{1}{r_t k_t + w_t h_t - k_{t+1}}\right) w_t = \gamma \frac{1}{1 - h_t}$$

$$\Rightarrow \quad 1 - h_t = \frac{y_t}{w_t} \gamma \left(1 - \frac{\beta B}{1 + \beta B}\right)$$

Given that $y_t = z_t k_t^\alpha h_t^{1-\alpha}$ and $w_t = (1-\alpha) z_t k_t^\alpha h_t^{-\alpha}$ we can rewrite the above as:

$$\begin{aligned} 1 - h_t &= \frac{z_t k_t^\alpha h_t^{1-\alpha}}{(1-\alpha) z_t k_t^\alpha h_t^{-\alpha}} \gamma \left(1 - \frac{\beta B}{1 + \beta B}\right) \\ &\Rightarrow \quad h_t = \bar{h} = \frac{(1-\alpha)(1+\beta B)}{(1-\alpha)(1+\beta B) + \gamma} \in (0, 1) \end{aligned}$$

Plug $\bar{h}$ into the first order condition for capital:

$$\begin{aligned} k_{t+1} &= y_t \left(\frac{\beta B}{1 + \beta B}\right) \\ &= z_t k_t^\alpha \bar{h}^{1-\alpha} \left(\frac{\beta B}{1 + \beta B}\right) \end{aligned}$$

Verify the guess for the value function:

$$\begin{aligned} A + B\ln(k_t) + C\ln(z_t) &= \underbrace{\ln\left[\underbrace{z_t k_t^\alpha \bar{h}^{1-\alpha}}_{r_t k_t + w_t h_t = y_t} - \underbrace{z_t k_t^\alpha \bar{h}^{1-\alpha} \left(\frac{\beta B}{1 + \beta B}\right)}_{k_{t+1}}\right]}_{\alpha \ln(k_t) + \ln\left(z_t \bar{h}^{1-\alpha}\left[1 - \frac{\beta B}{1+\beta B}\right]\right)} + \gamma \ln(1 - \bar{h}) + \beta A \\ &+ \underbrace{\beta B \ln\left(z_t k_t^\alpha \bar{h}^{1-\alpha} \frac{\beta B}{1 + \beta B}\right)}_{= \beta B \alpha \ln(k_t) + \beta B \ln(z_t \bar{h}^{1-\alpha} \frac{\beta B}{1+\beta B})} + \beta C E_t \ln(z_{t+1}) \\ &= (\alpha + \beta B \alpha) \ln(k_t) + \ln\left(z_t \bar{h}^{1-\alpha}\left[1 - \frac{\beta B}{1 + \beta B}\right]\right) + \gamma \ln(1 - \bar{h}) + \beta A \\ &+ \beta B \ln\left(z_t \bar{h}^{1-\alpha} \frac{\beta B}{1 + \beta B}\right) + \beta C E_t \ln(z_{t+1}) \end{aligned}$$

About the stochastic disturbance z_t. If we assume that $\{z\}$ can possibly take two values

$$A_{t+1} = \begin{cases} z^H & \text{with prob } p^H \\ z^L & \text{with prob } p^L \end{cases}$$

then the $E_{t+1} \ln(z_{t+1}) = p^H \ln(z^H) + p^L \ln(z^L) = \bar{z}$. In other words, it is just a constant.

We can now begin to match the coefficients in our guess with the coefficients in the above expression. In the case of the coefficient B:

$$\begin{aligned} B \ln(k_t) &= (\alpha + \beta B \alpha) \ln(k_t) \\ &\Rightarrow \quad B = \frac{\alpha}{1 - \beta\alpha} \end{aligned}$$

Replace the above into the first order condition for capital yields the policy function for capital:

$$\begin{aligned} k_{t+1} &= z_t k_t^{\alpha} \bar{h}^{1-\alpha} \left(\frac{\beta \left(\frac{\alpha}{1-\beta\alpha} \right)}{1 + \beta \left(\frac{\alpha}{1-\beta\alpha} \right)} \right) \\ &= z_t k_t^{\alpha} \underbrace{\bar{h}^{1-\alpha} \beta\alpha}_{\text{constant}} \end{aligned}$$

Replacing into the economy's resource constraint yields the policy function for consumption:

$$c_t = z_t k_t^{\alpha} \underbrace{\bar{h}^{1-\alpha} (1 - \beta\alpha)}_{\text{constant}}$$

which is quite similar to what we found for the deterministic version of the model, except that there are now two state variables in lieu of one. Identifying coefficients A and C are left as a practice exercise.

It is easy to see how for more complex models this method might not be very appealing as the algebra gets more challenging very fast. However, the method does provide a very elegant and tractable solution method for relatively small and simple models.