USC Econ 505 - Companion Notes

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Fall 2010

These notes only partially cover the first semester macro course at USC. The emphasis of the course will be in introducing models where consumer's behavior is derived as optimal responses given assumptions on preferences, production technology and uncertainty (i.e.: micro-founded models.). All throughout the course an agent's behavior is assumed to be rational: given the restrictions imposed by the primitives, all actors in the model are assumed to maximize their objective functions. Moreover, we will consider two market structures: a *time-0-trading* where all state-contingent claims ("Arrow-Debreu securities") are traded before any uncertainty is realized (i.e.: an options market); and a *sequential trading* where one-period state contingent insurance contracts ("Arrow securities") are traded each period. The notes follow closely the class syllabus and lectures rather than any one book in particular. Conceptually, some of the topics covered are:

- Competitive equilibrium (AD and SM) definition.
- Pareto Efficiency and the Social Planner formulation.
- 1^{st} and 2^{nd} welfare theorems.
- Nigishi's method for selecting the competitive allocations from the set of PE ones.
- Equivalence between the AD and the Sequential Markets equilibrium allocations.
- Consumption smoothing (see Appendix)
- Endowment and production economies

****DISCLAIMER**: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!

[I] Endowment Economies:

We begin with a very simple set-up in which there is no production and no uncertainty. This parsimonious economy is used to introduce several fundamental notions, like the concept of a competitive equilibrium, Pareto efficiency and consumption smoothing amongst others. In particular, we work with two market structures (Arrow-Debreu or time-zero and Sequential trading) which entail different assets and timings of trades but have identical equilibrium consumption allocations. Last, we describe a set-up in which agents must face uncertainty about their endowment streams. This economy is useful for studying risk sharing and asset pricing.

There are various reasons why agents will seek to trade with each other in these models, usually involving some kind of heterogeneity. For example, differences in preferences, initial endowments, risk tolerance or intertemporal discounting will all foster trade amongst agents. Particularly, in our deterministic set-up we assume an heterogeneity in the agents' endowment processes which will encourage them to trade in order to distribute consumption across time. In the stochastic version, agents face uncertainty about their endowment streams and seek to trade in an attempt to redistribute risk across the different states of nature. All in all, the the main insight provided by these models is that under a complete markets assumption, the agents' equilibrium allocations will always be a function of the realized aggregate endowment. In other words, each agent's consumption allocation will not depend on the particular realization of its individual endowment nor on the specific history leading up to that outcome but on the economy's total amount of available resources.

(a) The Deterministic set-up:

Time is discrete and indexed by $t = 0, 1, 2, ...$There are two individuals who live indefinitely in this pure exchange economy. The are no firms or government. There exist a single, non-storable consumption good. Individuals are assumed to have the following preferences over consumption:

$$U^i(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{1}$$

where $\beta \in (0,1)$ is the discount factor. There is no uncertainty in this model and both agents know their endowment patterns perfectly in advance. Moreover, all information is public.

Note that in a competitive equilibrium each agent in the economy (household or firm) is assumed to act independently from each other, seeking their own interest, and taking as given the fact that other agents will also seek their best. At an equilibrium all individual actions aggregate into the behavior of the whole economy. Aggregate demand will equal aggregate supply and the corresponding price will "clear the market".

We begin with an AD market structure where at period 0 before endowments are received and consumption takes place, the two agents meet and trade contracts for all future dates. Let p_t denote the price, in period 0, of one unit of consumption to be delivered in period t, in terms of an abstract unit of account. Both agents behave competitively in that they take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given and beyond their control when making their consumption decisions.

After trade has occurred, markets close and in all future periods the only thing that occurs is that agents meet and deliveries of the consumption goods take place. In other words, all trade takes place in period 0 and from then onwards agents become committed for all future periods to what they have agreed upon in the initial one.

With this set-up in mind we can define the AD competitive equilibrium:

Definition 1 *A competitive Arrow-Debreu equilibirum is a sequence of prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *such that:*

(i) Given prices $\{\hat{p}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{c}_t^i\right\}_{t=0}^{\infty}$ *solve the agents' optimization's problem:*

$$
\begin{aligned}
\max U^i &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t^i) && (2) \\
s.t. &: \sum\nolimits_{t=1}^{\infty} \hat{p}_t c_t^i \leq \sum\nolimits_{t=1}^{\infty} \hat{p}_t e_t^i \quad \forall i = 1,2 \\
&: c_t^i \geq 0 \ \forall t, \forall i = 1,2
\end{aligned}
$$

(ii) All markets clear:

$$
\sum\nolimits_{i=0}^{2} \hat{p}_t \hat{c}_t^i = \sum\nolimits_{i=0}^{2} \hat{p}_t e_t^i \quad \forall t \tag{3}
$$

Solving the model:

$$
\begin{aligned}
\max_{(\{\hat{c}_t^i\}_{t=0}^{\infty})_{u=1,2}} U^i &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t^i) && (4) \\
s.t. &: \sum\nolimits_{t=0}^{\infty} p_t c_t^i \leqslant \sum\nolimits_{t=0}^{\infty} p_t e_t^i && (5) \\
&: c_t^i \geqslant 0 \ \forall t && (6)
\end{aligned}
$$

Note there is only one budget constraint in the agents' problem. A lifetime or present value budget constraint. In turn, the lagrangean would be:

$$
L^i = \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t^i) + \lambda^i \left[{}_{t=0}^{\infty} p_t (e_t^i - c_t^i) \right] \tag{7}
$$

First order conditions:

$$
\begin{aligned}
[c_t^i] &: \beta^t u'(\hat{c}_t^i) = \lambda^i \hat{p}_t && (8) \\
[\lambda^i] &: {}_{t=0}^{\infty} \hat{p}_t (e_t^i - \hat{c}_t^i) = 0 && (9)
\end{aligned}
$$

Taking the ratio of 2 agents' FOCs yields:

$$\begin{aligned} \frac{\beta^t u'(\hat{c}_t^1)}{\beta^t u'(\hat{c}_t^2)} &= \frac{\lambda_1 \hat{p}_t}{\lambda_2 \hat{p}_t} \\ \frac{u'(\hat{c}_t^1)}{u'(\hat{c}_t^2)} &= \frac{\lambda_1}{\lambda_2} \end{aligned} \tag{10}$$

which implies that the ratio of marginal utilities will be constant across time. From above:

$$\Rightarrow \quad u'(\hat{c}_t^1) = \left(\frac{\lambda^1}{\lambda^2}\right) u'(\hat{c}_t^2) \tag{11}$$

$$\Rightarrow \quad \hat{c}_t^1 = u'^{-1}\left[\left(\frac{\lambda^1}{\lambda^2}\right) u'(\hat{c}_t^2)\right] \tag{12}$$

Replacing the above in the market clearing conditions yields:

$$\hat{c}_t^2 + u'^{-1}\left[\left(\frac{\lambda^1}{\lambda^2}\right) u'(\hat{c}_t^2)\right] = \sum\nolimits_{i=1}^{2} e_t^i \tag{13}$$

Example:

Assume agents face deterministic endowment streams $e^i = \left\{e_t^i\right\}_{t=0}^{\infty}$ of the consumption goods given by:

$$\begin{aligned} e_t^1 &= \left\{ \begin{array}{c} 2 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd} \end{array} \right\} \\ e_t^2 &= \left\{ \begin{array}{c} 0 \text{ if } t \text{ is even} \\ 2 \text{ if } t \text{ is odd} \end{array} \right\} \end{aligned} \tag{14}$$

Hence (13) would look like:

$$\hat{c}_t^2 + u'^{-1}\left[\left(\frac{\lambda_1}{\lambda_2}\right)u'(\hat{c}_t^2)\right] = 2 \ \forall t \tag{15}$$

which implies that $\hat{c}_t^1 \wedge \hat{c}_t^2$ will be constant across time. We'll denote them: $\bar{c}^1 \wedge \bar{c}^2$.

Solving for the equilibrium allocations:

Replacing the above into the FOCs:

$$\begin{aligned} \beta^t u'(\bar{c}^i) &= \lambda_i \hat{p}_t \\ &\Rightarrow \hat{p}_t = \frac{\beta^t u'(\bar{c}^i)}{\lambda_i} \quad \forall i = 1,2 \end{aligned} \tag{16}$$

Substitute (16) into agent's 1 budget constraint:

$$\begin{aligned} \sum\nolimits_{t=0}^{\infty} \frac{\beta^t u'(\bar{c}^1)}{\lambda_1}(e_t^1 - \bar{c}^1) &= 0 \\ &\Rightarrow \sum\nolimits_{t=0}^{\infty} \beta^t(e_t^1 - \bar{c}^1) = 0 \\ &\Rightarrow \sum\nolimits_{t=0}^{\infty} \beta^t e_t^1 = \bar{c}^1 \sum\nolimits_{t=0}^{\infty} \beta^t \\ &\Rightarrow \sum\nolimits_{t=0}^{\infty} \beta^t e_t^1 = \frac{\bar{c}^1}{1-\beta} \\ &= 2 + 2\beta^2 + 2\beta^4 + ... = \frac{\bar{c}^1}{1-\beta} \\ &= 2(\beta^0 + \beta^2 + \beta^4 + ...) = \frac{\bar{c}^1}{1-\beta} \\ &\Rightarrow \frac{2}{1-\beta^2} = \frac{\bar{c}^1}{1-\beta} \\ &\Rightarrow \bar{c}^1 = 2\left[\frac{1-\beta}{1-\beta^2}\right] = 2\left[\frac{1-\beta}{(1-\beta)(1+\beta)}\right] \\ &\Rightarrow \bar{c}^1 = \frac{2}{1+\beta} > 1 \quad \forall t \\ &\Rightarrow \bar{c}^2 = 2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta} < 1 \quad \forall t \end{aligned}$$

Hence, the two agents differ only in their respective wealth. Because agent 1 receives the endowment first and since both agents are impatient, agent 1 will be relatively better off than agent 2. This feature is captured by the fact that agent's 1 lifetime utility will be higher than agent 2.

Finally, to complete the equilibrium we need to find the equilibrium prices:

Solving for the equilibrium prices:

From the first order conditions, we have that at time "t" and "$t+1$" agents solve:

$$\begin{aligned} \frac{\beta^t}{\hat{c}_t^i} &= \lambda^i \hat{p}_t \quad \forall i = 1,2 && (17) \\ \frac{\beta^{t+1}}{\hat{c}_{t+1}^i} &= \lambda^i \hat{p}_{t+1} \quad \forall i = 1,2 && (18) \end{aligned}$$

$$\Rightarrow \hat{p}_{t+1}\hat{c}_{t+1}^i = \beta \hat{p}_t \hat{c}_t^i \quad \forall t, \forall i = 1,2 \tag{19}$$

Summing across agents yields:

$$\hat{p}_{t+1}\left[\hat{c}_{t+1}^1 + \hat{c}_{t+1}^2\right] = \beta \hat{p}_t \left[\hat{c}_t^1 + \hat{c}_t^2\right] \tag{20}$$

Using the market clearing conditions yields:

$$\begin{aligned} \hat{p}_{t+1}\left[e_{t+1}^1 + e_{t+1}^2\right] &= \beta \hat{p}_t \left[e_t^1 + e_t^2\right] && (21) \\ &\Rightarrow \hat{p}_{t+1} = \beta \hat{p}_t \\ \text{Solving recursively} &: \\ \hat{p}_{t+1} &= \beta^{t+1} \hat{p}_0 \end{aligned}$$

Making consumption at $t = 0$ the numeraire good (ie: $p_0 = 1$), yields the equilibrium condition that needs to be satisfied:

$$\hat{p}_t = \beta^t \tag{22}$$

Since $\beta < 1$, the period 0 price for period t consumption is lower than the period 0 price for period 0 consumption. In other words, in this economy prices are declining over time reflecting the agents' impatience.

About the gains from trade:

It is easy to see how both agents benefit from trading at $t = 0$. Under trade, their respective lifetime utility would be:

$$U^1 = \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2}{1+\beta}\right) = \frac{\ln\left(\frac{2}{1+\beta}\right)}{1-\beta} > 0 \tag{23}$$

$$U^2 = \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2\beta}{1+\beta}\right) = \frac{\ln\left(\frac{2\beta}{1+\beta}\right)}{1-\beta} < 0 \tag{24}$$

(note: that utility is nothing than an ordinal measure. The fact that is negative for agent 2 only matter for ordering/comparison purposes.)

Under autarky conditions (ie: no trade is possible), both agents would receive a lifetime utility of:

$$U^i = -\infty \quad \forall i = 1, 2 \tag{25}$$

Therefore, both agents are better off trading than just consuming their own endowment streams. In this sense, since both agents are better off, the equilibrium can be thought of being socially optimal. For a description on the Negishi method to compute competitive equilibria see the Apprendix.

(b) The Stochastic set-up:

Let s^t be an event history up to period t, where $s^t \in S^t$

Let $p_t(s^t)$ denote the price of one unit of consumption quoted at period 0, delivered at period t iff history s^t has been realized.

With the above in mind, it is possible to define the agents' problem as:

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} U^i = \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) \tag{26}$$

$$s.t. \quad : \quad \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leqslant \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) e_t^i(s^t) \tag{27}$$

$$: \quad c_t^i(s^t) \geqslant 0 \quad \forall t, \forall s^t \in S^t \tag{28}$$

Lagrangean set-up:

$$L^i = \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) + \lambda_i \left[\sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) e_t^i(s^t) - p_t(s^t) c_t^i(s^t) \right] \tag{29}$$

First order conditions:

$$[c_t^i] \quad : \quad \beta^t \pi(s^t) u'(\hat{c}_t^i(s^t)) = \lambda_i \hat{p}_t(s^t) \quad \forall t, \forall i, \forall s^t \in S^t \tag{30}$$

$$[\lambda_i] \quad : \quad \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) \left[e_t^i(s^t) - \hat{c}_t^i(s^t) \right] = 0 \quad \forall t, \forall i, \forall s^t \in S^t \tag{31}$$

Taking the ratio of the two agent's FOCs w.r.t to c_t yields:

$$\frac{\beta^t \pi(s^t) u'(\hat{c}_t^i(s^t))}{\beta^t \pi(s^t) u'(\hat{c}_t^j(s^t))} = \frac{\lambda_i \hat{p}_t(s^t)}{\lambda_j \hat{p}_t(s^t)} \tag{32}$$
$$\Rightarrow \frac{u'(\hat{c}_t^i(s^t))}{u'(\hat{c}_t^j(s^t))} = \frac{\lambda_i}{\lambda_j}$$

Note that (32) implies that the ratios of marginal utilities between pairs of agents will be constant across all histories and dates.

Solving for $\hat{c}_t^j(s^t)$ from above yields the relationship between the consumption of agent j relative to agenti i:

$$\hat{c}_t^j(s^t) = u'^{-1}\left[u'(\hat{c}_t^i(s^t)\frac{\lambda_j}{\lambda_i}\right] \tag{33}$$

Substitute (33) into the market clearing condition to obtain (assuming there are only two agents i and j):

$$\hat{c}_t^i(s^t) + u^{'-1}\left[u'(\hat{c}_t^i(s^t)\frac{\lambda_j}{\lambda_i}\right] = \sum_{i=1}^{2} e_t^i(s^t) \quad \forall t, \forall i, \forall s^t \in S^t \tag{34}$$

Note the right hand side of (34) is the current realization of the aggregate endowment. Therefore it follows that the equilibrium allocation $\hat{c}_t^i(s^t)$ for each i depends only on the economy's aggregate endowment.

Example:

Consider an AD market structure where the random variable s_t is *iid* across time and takes value of 1 and 0 with probabilities 0.6 and 0.4 respectively. If $s_t = 0$ the agent i gets an endowment of 2 and agent j gets 0. If $s_t = 1$ then agent i is endowed with 0 and agent j with 2. Note the constant aggregate endowment.

Then from (34) we get that:

$$\hat{c}_t^i(s^t) + u^{'-1}\left[u'(\hat{c}_t^i(s^t)\frac{\lambda_j}{\lambda_i}\right] = 2 \quad \forall t, \forall i, \forall s^t \in S^t$$

which in turn implies that $c_t^i(s^t)$ and $c_t^j(s^t)$ will be fixed across times and histories. We call these quantities $\bar{c}^i$ and $\bar{c}^j$:

$$\Rightarrow \quad c_t^i(s^t) = \bar{c}^i \quad \forall t, \forall s^t \in S^t$$
$$\Rightarrow \quad c_t^j(s^t) = \bar{c}^j \quad \forall t, \forall s^t \in S^t$$

Moreover, from (30) we can pin down the price level:

$$\hat{p}_t(s^t) = \frac{\beta^t \pi(s^t) u'(\bar{c}^i)}{\lambda_i} \tag{35}$$

Substitute the above into the FOC:

$$\begin{aligned}
\sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t)[e_t^i(s^t) - \bar{c}_t^i] &= 0 \qquad (36)\\
&\Rightarrow \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \frac{\beta^t \pi(s^t) u'(\bar{c}_t^i)}{\lambda i}[e_t^i(s^t) - \bar{c}^i] = 0\\
&= \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t)[e_t^i(s^t) - \bar{c}^i] = 0\\
&= \sum\nolimits_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_t^i(s^t) = \frac{\bar{c}^i}{1-\beta}
\end{aligned}$$

For agent i, given that:

$P(s_t = 1) = 0.6 \Rightarrow e = 2$

$P(s_t = 0) = 0.4 \Rightarrow e = 1$

$$
\begin{aligned}
\textstyle\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_t^i(s^t) &= 2 * 0.6 + 2 * \beta 0.6 + ... \\
&= (2 * 0.6) \left[1 + \beta + \beta^2 + \beta^3 + ... + \beta^{\infty}\right] \\
&= \frac{2 * 0.6}{1-\beta} = \frac{1.2}{1-\beta} \\
&\Rightarrow \frac{1.2}{1-\beta} = \frac{\bar{c}^i}{1-\beta} \; (agent \; i) \\
&\Rightarrow \bar{c}^i = 1.2 \; \forall t, \forall s^t \in S^t \\
&\Rightarrow \bar{c}^j = 0.8 \; \forall t, \forall s^t \in S^t
\end{aligned}
$$

Note that under complete markets, each agent's consumption will be a constant proportion of the economy's total amount of resources. In turn, if the aggregate endowment is constant across time and all states, then the equilibrium consumption allocations will also be constant across time and all states. Recall that in a deterministic environment (and a set up where agents benefit from trading) individuals would consume a constant amount each period in equilibrium. As such in a deterministic set up, agents consume constant quantities in equilibrium while in a stochastic one they consume constant fractions of the realized aggregate endowment. This will enable us to use a representative agent framework not because all agents are equal, but because by studying the representative agent we can re-construct the aggregate. Finally, from the aggregate we could calculate each agent's equilibrium allocation.

[II] Production Economies: (deterministic and stochastic)

The purpose of this section is to provide several examples on how to set-up an optimization problem, identify the set of equations characterizing the equilibrium as well as defining several of them under different environments. Some of the topics covered are:

1) Sequential formulation:

- A.D. market structure
- Sequential Markets market structure
- Social Planner set-up

2) Recursive formulation:

- Social Planner set-up
- Recursive Competitive Equilibrium (RCE)

To exemplify the above, I'll use a few different model specifications outlined below.

General set-up description:

(I) There are a large number of identical households (HH) with preferences $u(c_t, l_t)$, where u is continuous, increasing, concave and differentiable in both its arguments. Here c_t and l_t will stand for consumption and leisure in period t respectively. There is one single firm which produces goods using a constant returns to scale (CRT) technology $F(k_t, n_t)$, where k_t is capital and n_t is hours worked.

HH carry out the capital accumulation in the economy and rent our their capital and labour services to firms. They are endowed with $k_0 > 0$ units of capital in period 0. Capital is assumed to depreciate at the rate δ in each period. Moreover, HH are endowed with one unit of time that can be allocated either to leisure or work. HH also purchase output from the firm and use it for consumption or store it as capital which can be used in the following period.

Last, in the stochastic version, the production technology is now subject to a TFP shock (z_t) every period. Agents observe the realization of z_t before making their optimal choices.

(II) The same as part (I) except that now there are two firms, one producing capital goods while the other produces consumption ones. Both firms employ labour and capital supplied by the HH to produce their output and both factors of production are perfectly substitutable and free to move from one sector to the other. In the version with uncertainty, firms in each sector are subject to an idiosyncratic iid productivity shock.

(III) The same economy as described in part (II) except the capital used by firms is sector specific (i.e.: the capital used by one firm is not substitutable by the capital used by the other firm). Labor is free to move across sectors. Following (II), the stochastic version of this problem features idiosyncratic shocks with the main difference being that capital is no longer mobile across sectors.

The following table summarizes all cases (deterministic and stochastic) under consideration:

Model	Formulation	Centralized Solution	Decentralized Solution	
		Social Planner	AD mkt struct	Seq mkt struct
(I)	Sequential	(1)	(3)	(4)
	Recursive	(2)	(5)	
(II)	Sequential	(6)	(8)	(9)
	Recursive	(7)	(10)	
(III)	Sequential	(11)	(13)	(14)
	Recursive	(12)	(15)	

Note that throughout we will assume that factor markets behave competitively. This assumption, together with the constant return to scale of the production technology used by firms implies that the latter will (in equilibrium) make no profits and so their ownership needs not to be specified.

This, however, will change if the production technology is not CRT or factor markets do not behave competitively.

About the nature of uncertainty

In all the models below, output is produced combining capital and labor employing a standard neoclassical production technology : $z_t F(k_t, n_t)$. Note that the nature of $\{z_t\}_{t=0}^{\infty}$ can accommodate many different kinds of specifications. Succinctly:

- If $\{z_t\}$ is known or constant, then the model is said to be deterministic.
- If $\{z_t\}$ follows some random process, then the model is said to be stochastic

With this in mind, in the deterministic models below it will be assumed that $\{z_t\} = 1$ for all t. If on the other hand the model is stochastic, $\{z_t\}$ will be assumed to follow some random process which shall be specified in the problem's set-up. There is more information about modeling stochastic shocks in the Appendix.

Model I

(1) Social Planner + Sequential formulation[1]**:**

(a) Deterministic:

The solution to the SP problem is the set of PE allocations. Note that under a representative agent framework there are no weights in the P.O. problem set-up, since the social planner only needs to care about 1 agent's utility (i.e.: the representative one).

Planner's problem :

$$\begin{aligned} \max_{\{c_t,i_t,l_t\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad (37) \\ s.t. &: c_t + i_t \leq F(k_t, n_t) \text{ (resource/feasibility constraint)} \\ &: k_{t+1} = (1-\delta)k_t + i_t \text{ (law of motion of capital)} \\ &: n_t + l_t = 1 \text{ (time endowment)} \\ &: c_t, k_{t+1} \geq 0 \ \forall t \text{ (non-negativity condition)} \\ &: k_0 > 0 \text{ given} \end{aligned}$$

The lagrangean for this problem would be:

$$L = \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1-n_t) + \lambda_t [F(k_t, n_t) - c_t - k_{t+1} + (1-\delta)k_t]\} \qquad (38)$$

Note that the non-negativity constraint on c_t is not included in the lagrangean because it will never be binding given the properties assumed for the utility function. Similarly, the non-negativity condition on the capital stock is also often excluded as it will never be binding given the properties assumed for the production function (see Appendix for properties of the utility and production functions).

[1] Note that since there are no markets in a social planner set-up, we'll refer to sequential and recursive "formulations" of the problem.

Last, note that the time endowment as well as the accumulation of capital (both equality constraints) were substituted into the lagrange objective function. In doing so, we've replaced i_t and l_t for k_{t+1} and n_t as choice variables.

Assuming an interior solution, the first order conditions of the problem are[2]:

$$\left[\frac{\delta L}{\delta c_t}\right] = 0 \Leftrightarrow u_c(c_t, 1-n_t) = \lambda_t \tag{39}$$

$$\left[\frac{\delta L}{\delta k_{t+1}}\right] = 0 \Leftrightarrow \beta\lambda_{t+1}\left[F_k(k_{t+1}, n_{t+1}) + (1-\delta)\right] = \lambda_t \tag{40}$$

$$\left[\frac{\delta L}{\delta n_t}\right] = 0 \Leftrightarrow u_l(c_t, 1-n_t) = \lambda_t F_n(k_t, n_t) \tag{41}$$

$$\left[\frac{\delta L}{\delta \lambda_t}\right] = 0 \Leftrightarrow F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t \tag{42}$$

After some substitution, every period we are left with three unknowns c_t, n_t, k_{t+1} and three equations characterizing the optimal solution:

(39) + (40):

$$\beta u_c(c_{t+1}, 1-n_{t+1})\left[F_k(k_{t+1}, n_{t+1}) + (1-\delta)\right] = u_c(c_t, 1-n_t) \tag{43}$$

(39) + (41):

$$u_l(c_t, 1-n_t) = u_c(c_t, 1-n_t)F_n(k_t, n_t) \tag{44}$$

(42):

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t \tag{45}$$

Note that (43) is the intertemporal optimality condition for consumption, usually referred to as Euler equation. This equation has an intuitive economic interpretation: along an optimal path the

[2] In turn, every period we have 4 endogenous variables c_t, n_t, λ_t, k_{t+1} and 4 equations (39) - (42).

marginal utility from consumption at any point in time is equal to its opportunity cost. In addition, (44) is the intratemporal condition for leisure-consumption choice. (45) is the economy's resource constraint.

Definition 2 *An allocation* $\{c_t, i_t, l_t\}_{t=0}^{\infty}$ *is Pareto efficient if it is feasible and there exist not an alternative feasible allocation* $\left\{\hat{c}_t, \hat{\imath}_t, \hat{l}_t\right\}_{t=0}^{\infty}$ *such that:*

$$u(\hat{c}_t, \hat{l}_t) > u(c_t, l_t)$$

(b) Stochastic:

A brief word on notation.

Let $s_t \in S$ denote a particular event realization at time t

Let $S = \{\eta_1, \eta_2, ..., \eta_N\}$ the set of all possible events that can occur in period t. This set is not only assumed to be finite, but also to be constant for every period.

Let s^t denote a particular history of events up to time t; i.e.: $s^t = (s_0, s_1, ..., s_t)$. It is a vector of length $t+1$ that summarizes the particular realization of all events up to t.

Let $S^t = S \times S \times ... \times S \Rightarrow s^t \in S^t$ be the set of all possible histories up to time t

Let $\pi(s^t)$ denote the probability of a particular event history occurring.

It is assumed that $\pi(s^t) > 0 \ \forall s^t \in S^t$

Given this notation we'll have that lifetime HH preferences will be represented as:

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t)) \\ \text{or, alternatively} &: \\ U &= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \end{aligned}$$

Note that under the second notation type, we need not to specify the dependance on the event history s^t.

Planner's problem now becomes :

$$\begin{aligned}
\max_{\{c_t,i_t,l_t\}_{t=0}^{\infty}} U &= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) && (46)\\
s.t. &: c_t + i_t \leq z_t F(k_t, n_t) \text{ (resource/feasibility constraint)}\\
&: k_{t+1} = (1-\delta)k_t + i_t \text{ (law of motion of capital)}\\
&: n_t + l_t = 1 \text{ (time endowment)}\\
&: c_t, k_{t+1} \geq 0 \ \forall t \text{ (non-negativity condition)}\\
&: \ln z_{t+1} = \rho_z \ln z_t + \epsilon_{t+1} \text{(law of motion of TFP shock)}\\
&: k_0 > 0, z_0 \text{ given}
\end{aligned}$$

The lagrangean for this problem would be:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1-n_t) + \lambda_t [z_t F(k_t, n_t) - c_t - k_{t+1} + (1-\delta)k_t] \right\} \tag{47}$$

With similar first order conditions to part (a), except that now z_t will be random and every period it will affect the marginal product of capital and labor.

(2) Social Planner + Recursive formulation

(a) Deterministic:

Planner's problem:

$$
\begin{aligned}
V(K) &= \max_{c,l,i} \{u(c,l) + \beta V(K')\} && (48)\\
s.t. &: \; c + i \leq F(K,n) \text{ (resource/feasibility constraint)}\\
&: \; K' = (1-\delta)K + i \text{ (law of motion of capital)}\\
&: \; n + l = 1 \text{ (time endowment)}\\
&: \; c, K' \geq 0 \text{ (non-negativity condition)}\\
&: \; k_0 > 0 \text{ given}
\end{aligned}
$$

(b) Stochastic:

Planner's problem:

$$
\begin{aligned}
V(K,z) &= \max_{c,l,i} \{u(c,l) + \beta E V(K',z')\} && (49)\\
s.t. &: \; c + i \leq zF(K,n) \text{ (resource/feasibility constraint)}\\
&: \; K' = (1-\delta)K + i \text{ (law of motion of capital)}\\
&: \; n + l = 1 \text{ (time endowment)}\\
&: \; z' = \rho_z z + \epsilon' \text{ (law of motion of TFP shock)}\\
&: \; c, K' \geq 0 \text{ (non-negativity condition)}\\
&: \; k_0 > 0, z_0 \text{ given}
\end{aligned}
$$

where E denotes the expectation conditional on the optimizing agent's information set at time t.

(3) Decentralized + A.D. mkts

(a) Deterministic:

HH's problem:

$$
\begin{aligned}
\max_{\{c_t, i_t, n_t^s, k_t^s\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t, l_t) && (50) \\
s.t. &: \sum\nolimits_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum\nolimits_{t=0}^{\infty} p_t(w_t n_t^s + r_t k_t^s + \pi_t) \text{ (PVBC)} \\
&: l_t + n_t^s = 1 \text{ (time endowment)} \\
&: x_{t+1} = (1-\delta)x_t + i_t \text{ (law of motion of capital)} \\
&: k_t^s \leq x_t, \forall t \\
&: c_t, x_{t+1} \geq 0 \ \forall t \\
&: x_0 > 0 \text{ given}
\end{aligned}
$$

where p_t represents the time zero price of a consumption good delivered in period t and w_t and r_t are the rental rates to be paid in period t. Also k_t^s stands for capital services and x_t for the capital stock in period t.

The associated lagrangean would be:

$$
L = \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t^s) + \lambda \left[\sum_{t=0}^{\infty} p_t(w_t n_t^s + r_t k_t^s + \pi_t - c_t - k_{t+1} + (1-\delta)k_t) \right] \right\} \quad (51)
$$

Firm's problem:

$$
\begin{aligned}
\max_{\left\{n_t^d, k_t^d\right\}_{t=0}^{\infty}} \pi_t &= \sum\nolimits_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t n_t^d) && (52) \\
s.t. &: y_t = F(k_t^d, n_t^d) \\
&: y_t, k_t^d, n_t^d \geq 0 \ \forall t
\end{aligned}
$$

Definition 3 *A competitive AD equilibrium is a sequence of prices* $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *and the HH* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t\right\}_{t=0}^{\infty}$ *such that:*

(i) Given prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t\right\}_{t=0}^{\infty}$ *solve the HH optimization's problem.*

(ii) Given prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *solve the firm's optimization's problem.*

(iii) All markets clear:

$\hat{k}_t^s = \hat{k}_t^d$ *(capital services market)*

$\hat{n}_t^s = \hat{n}_t^d$ *(labour market)*

$\hat{y}_t = \hat{c}_t + \hat{\imath}_t$ *(goods market)*

(b) Stochastic:

To be completed ...

(4) Decentralized + Sequential mkts

(a) Deterministic:

$HH's$ sequential problem:

$$\begin{aligned}
\max_{\{c_t,i_t,n_t^s,k_t^s,b_{t+1}\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t, l_t) && (53)\\
s.t. &: c_t + i_t + q_t b_{t+1} \leq w_t n_t^s + r_t k_t^s + \pi_t + b_t \text{ (period budget constraint)}\\
&: x_{t+1} = (1-\delta)x_t + i_t \text{ (law of motion of capital)}\\
&: l_t + n_t^s = 1 \text{ (time endowment)}\\
&: k_t^s \leq x_t \ \forall t\\
&: q_t = \frac{1}{r_{t+1}}\\
&: b_{t+1} \geq -\bar{B} \text{ (no Ponzi game condition)}\\
&: c_t, x_{t+1} \geq 0 \ \forall t \ \text{ (non-negativity constraint)}\\
&: x_0 > 0 \text{ and } b_0 \geq 0 \text{ given}
\end{aligned}$$

Firm's problem:

$$\begin{aligned}
\max_{\{y_t,n_t^d,k_t^d\}_{t=0}^{\infty}} \pi_t &= y_t - r_t k_t^d - w_t n_t^d && (54)\\
s.t. &: y_t = F(k_t^d, n_t^d)\\
&: y_t, k_t^d, n_t^d \geq 0 \ \forall t
\end{aligned}$$

The household's associated lagrangean would be:

$$L = \sum_{t=0}^{\infty} \left\{\beta^t u(c_t, 1-n_t^s) + \lambda_t[w_t n_t^s + r_t k_t^s + b_t + \pi_t - c_t - x_{t+1} + (1-\delta)x_t - q_t b_{t+1}]\right\} \quad (55)$$

Note that in building the Lagrangean one needs to keep in mind the same considerations in terms of non-negativity constraints as in (47). Additinally, we have that in equilibrium:

i) due to the constant returns to scale technology assumption on the production function $\pi_t = 0 \; \forall t$

ii) the no-Ponzi game condition will never bind.

iii) due to the monotonicity of preferences the agent will always find it profitable to rent out all of its available capital. In turn $k_t^s = x_t = k_t \; \forall t$

With this in mind we can re-write (55) as:

$$L = \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t, 1 - n_t^s) + \lambda_t [w_t n_t^s + r_t k_t + b_t - c_t - k_{t+1} + (1-\delta) k_t - q_t b_{t+1}] \right\} \tag{56}$$

The HH's original choice variables were $(c_t, i_t, b_{t+1}, k_t^s, n_t^s)$. After our assumptions and substitutions they have become $(c_t, k_{t+1}, b_{t+1}, n_t^s)$.

Moreover, replacing the technological constraint into the firm's objective function, we can write its optimization problem as:

$$\max_{\left\{n_t^d, k_t^d\right\}_{t=0}^{\infty}} \pi_t = F(k_t^d, n_t^d) - r_t k_t^d - w_t n_t^d \tag{57}$$

$$\tag{58}$$

The HH's FOC are:

$$\left[\frac{\delta L}{\delta c_t}\right] = 0 \Leftrightarrow u_c(c_t, 1 - n_t^s) = \lambda_t \tag{59}$$

$$\left[\frac{\delta L}{\delta n_t^s}\right] = 0 \Leftrightarrow u_l(c_t, 1 - n_t^s) = \lambda_t w_t \tag{60}$$

$$\left[\frac{\delta L}{\delta k_{t+1}}\right] = 0 \Leftrightarrow \lambda_t = \beta \lambda_{t+1} [(1+\delta) + r_{t+1}] \tag{61}$$

$$\left[\frac{\delta L}{\delta b_{t+1}}\right] = 0 \Leftrightarrow \beta\lambda_{t+1} = q_t\lambda_t \tag{62}$$

$$\left[\frac{\delta L}{\delta \lambda_t}\right] = 0 \Leftrightarrow w_t n_t^s + r_t k_t + b_t = c_t + k_{t+1} - (1-\delta)k_t + q_t b_{t+1} \tag{63}$$

The firm's FOC are:

$$\left[\frac{\delta \pi}{\delta k_t^d}\right] = 0 \Leftrightarrow F_k(k_t^d, n_t^d) = r_t \tag{64}$$

$$\left[\frac{\delta \pi}{\delta n_t^d}\right] = 0 \Leftrightarrow F_n(k_t^d, n_t^d) = w_t \tag{65}$$

Note that the HH solves a dynamic optimization problem given the capital accumulation problem it faces. The firms, on the other hand, solve a static optimization problem. Every period we have 4 unknowns $(c_t, k_{t+1}, b_{t+1}, n_t)$ and four equations (66) - (69):

(59) + (60) + (65):

$$u_l(c_t, 1-n_t^s) = u_c(c_t, 1-n_t^s)F_n(k_t^d, n_t^d) \tag{66}$$

(59) + (61) + (64):

$$u_c(c_t, 1-n_t^s) = \beta u_c(c_{t+1}, 1-n_{t+1}^s)[(1+\delta) + F_k(k_{t+1}^d, n_{t+1}^d)] \tag{67}$$

(59) + (62):

$$\beta u_c(c_{t+1}, 1-n_{t+1}^s) = q_t u_c(c_t, 1-n_t^s) \tag{68}$$

(63) + (64) + (65):

$$F_n(k_t^d, n_t^d)w_t + F_k(k_t^d, n_t^d)r_t + b_t = c_t + k_{t+1} - (1-\delta)k_t + q_t b_{t+1} \tag{69}$$

Definition 4 *A competitive sequential markets equilibrium is a sequence of prices* $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *and the HH* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t, \hat{b}_{t+1}\right\}_{t=0}^{\infty}$ *such that:*

(i) Given prices $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t, \hat{b}_{t+1}\right\}_{t=0}^{\infty}$ *solve the HH optimization's problem.*

(ii) Given prices $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *solve the firm's optimization's problem.*

(iii) All markets clear:

$\hat{k}_t^s = \hat{k}_t^d$ *(capital services market)*

$\hat{n}_t^s = \hat{n}_t^d$ *(labour services market)*

$\hat{y}_t = \hat{c}_t + \hat{\imath}_t + \hat{b}_{t+1}$ *(goods market)*

$\hat{b}_{t+1} = 0$ *(asset market)*

(b) Stochastic:

The representative household now faces uncertainty about the realization of the productivity shock. Being risk adverse the agent will seek to protect itself against a potentially adverse outcome by acquiring a portfolio of assets that deliver a unit of consumption goods in each particular state of the world (each asset will only pay in a specific state). If there are as many assets as possible states of nature, the financial market is said to be complete.

HH's sequential problem [3]:

$$
\begin{aligned}
\max_{\{c_t, i_t, n_t^s, k_t^s, b_{t+1}\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad (70)\\
s.t. &: c_t + i_t + \sum_{s_{t+1} \in S} q_t b_{t+1} \leq w_t n_t^s + r_t k_t^s + b_t + \pi_t \\
&: x_{t+1} = (1-\delta) x_t + i_t \\
&: l_t + n_t^s = 1 \\
&: k_t^s \leq x_t \ \forall t \\
&: q_t = \frac{1}{r_{t+1}} \\
&: b_{t+1} \geq -\bar{B} \\
&: c_t, x_{t+1} \geq 0 \ \forall t \\
&: x_0 > 0 \text{ and } b_0 \geq 0 \text{ given}
\end{aligned}
$$

The firm's problem is now:

$$
\begin{gathered}
\max_{\{n_t^d, k_t^d\}_{t=0}^{\infty}} z_t F(k_t^d, n_t^d) - r_t k_t^d - w_t^d \qquad (71)\\
s.t. : \ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1}
\end{gathered}
$$

[3] Note that, technically, every price and variable will always be a function of the realized history of shocks in this set-up. For example, the budget constraint should look something like: $w_t(s^t) n_t^s(s^t) + r_t(s^t) k_t^s(s^t) s + b_t(s^t) + \pi_t(s^t) + \geq c_t(s^t) + i_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) b_{t+1}(s^t, s_{t+1})$. The price $q_t(s^t)$ can be interpreted as the price in period t and event history (s^t) for buying one unit of consumption delivered for sure in period $t+1$. In a slight abuse of notation I've remove the history dependance. Hence $q_t(s^t)$ will be written as q_t and so on.

The problem's lagrangean is:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1 - n_t^s) + \lambda_t [w_t n_t^s + r_t k_t + b_t - c_t - k_{t+1} + (1-\delta)k_t - \sum q_t b_{t+1}] \right\}$$

The first order conditions are:

$$\begin{aligned}
\frac{\delta L}{\delta c_t} = 0 \quad &\Longleftrightarrow \quad u_c(c_t, 1 - n_t) = \lambda_t \\
\frac{\delta L}{\delta n_t} = 0 \quad &\Longleftrightarrow \quad u_l(c_t, 1 - n_t) = \lambda_t w_t \\
\frac{\delta L}{\delta k_{t+1}} = 0 \quad &\Longleftrightarrow \quad \beta E_t \lambda_{t+1} [r_{t+1} + 1 - \delta] = \lambda_t \\
\frac{\delta L}{\delta b_{t+1}} = 0 \quad &\Longleftrightarrow \quad \beta E_t (\lambda_{t+1} q_{t+1}) = \lambda_t
\end{aligned}$$

It is important to remember that these FOCs must hold for all times t and for all histories s^t.

(5) Decentralized + Recursive formulation (RCE)

(a) Deterministic:

Let K represent the aggregate capital stock in the economy

Let k represent the individual capital stock of the household

Let c be the numeraire good making $r(K)$ and $w(K)$ the rental prices for capital and labor in the economy in terms of the consumption good.

The HH problem:

$$\begin{aligned} V(K,k) &= \max_{c,i,l}\{u(c,l)+\beta V(K',k')\} \qquad (72) \\ s.t. &: \quad c+i \leq r(K)k+w(K)n+\pi \\ &: \quad k'=(1-\delta)k+i \text{ (law of motion for individual capital)} \\ &: \quad l+n=1 \text{ (time constraint)} \\ &: \quad c,k' \geq 0 \ \forall t \geq 0 \text{ (non-negativity)} \\ &: \quad K'=G(K) \text{ (perceived law of motion of the aggregate state)} \end{aligned}$$

Firm's problem:

$$\max_{k^d,n^d} \pi = F(k^d,n^d) - w(K)n^d - r(K)k^d \qquad (73)$$

Definition 5 *A Recursive Competitive Equilibrium (RCE) for this economy consists of the following* *functions:*

a) A value function $V(K,k)$ *and decision rules for the* HH $c(K,k), l(K,k)$ *and* $i(K,k)$

b) Decision rules for the firm $k^d(K)$ *and* $n^d(K)$

c) Price function $w(K)$ *and* $r(K)$

d) A perceived law of motion for the aggregate state for the HH, $K' = G(K)$

such that :

i) Given c) and d), a) solves the HH *optimization problem*

ii) Given c), b) solves the firm's optimization problem

iii) All markets clear:

$n^d(K) = n$

$k^d(K) = k$

$F(k^d, n^d) = c + i$

iv) Perceptions are correct: $G(K) = k'(K,k) = k'(K,K)$

(b) Stochastic:

The HH problem:

$$\begin{aligned} V(K,k,z) &= \max_{c,i,l} \{u(c,l) + \beta E V(K',k',z')\} \\ s.t. &: c + i \leq r(K,z)k + w(K,z)n + \pi \\ &: k' = (1-\delta)k + i \text{ (law of motion for individual capital)} \\ &: l + n = 1 \text{ (time constraint)} \\ &: \ln z' = \rho \ln z + \epsilon' \text{ (Law of motion of TFP shock)} \\ &: c, k' \geq 0 \ \forall t \geq 0 \text{ (non-negativity)} \\ &: K' = G(K,z) \text{ (perceived law of motion of the aggregate state)} \end{aligned} \tag{74}$$

Firm's problem:

$$\max_{k^d, n^d} \pi = zF(k^d, n^d) - w(K,z)n^d - r(K,z)k^d \tag{75}$$

Definition 6 *A Recursive Competitive Equilibrium (RCE) for this economy consists of the following* *functions:*

a) A value function $V(K,k,z)$ *and decision rules for the* HH $c(K,k,z), l(K,k,z)$ *and* $i(K,k,z)$

b) Decision rules for the firm $k^d(K,z)$ *and* $n^d(K,z)$

c) Price function $w(K,z)$ *and* $r(K,z)$

d) A perceived law of motion for the aggregate state for the HH, $K' = G(K,z)$

such that :

i) Given c) and d), a) solves the HH *optimization problem*

ii) Given c), b) solves the firm's optimization problem

iii) All markets clear:

$n^d(K,z) = n$

$k^d(K,z) = k$

$F(k^d, n^d) = c + i$

iv) Perceptions are correct: $G(K,z) = k'(K,k,z) = k'(K,K,z)$

Model II

For models II and III I will just elaborate on the deterministic set-up. The stochastic case closely follows the examples developed for model I.

(6) Social Planner + Sequential Formulation

Planner's problem :

$$\begin{aligned}
\max_{\{c_t,i_t,n_t^1,n_t^2,k_t^1,k_t^2\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad (76) \\
s.t. &: \quad c_t \leq F^1(k_t^1, n_t^1) \text{ (consumer goods feasibility constraint)} \\
&: \quad i_t \leq F^2(k_t^2, n_t^2) \text{ (capital goods feasibility constraint)} \\
&: \quad k_{t+1} = (1-\delta)k_t + i_t \text{ (law of motion of capital)} \\
&: \quad k_t^1 + k_t^2 = k_t \text{ (economy wide capital)} \\
&: \quad n_t^1 + n_t^2 + l_t = 1 \text{ (time endowment)} \\
&: \quad c_t, k_{t+1} \geq 0 \ \forall t \text{ (non-negativity condition)} \\
&: \quad k_0 > 0 \text{ given}
\end{aligned}$$

The lagrangean for this problem would be:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1 - n_t^1 - n_t^2) + \lambda_t [F^1(k_t^1, n_t^1) - c_t] + q_t [F^2(k_t - k_t^1, n_t^2) - k_{t+1} - (1-\delta)k_t] \right\} \qquad (77)$$

where λ_t and q_t are the lagrange multipliers for the consumer goods and capital goods feasibility constraints respectively.

The first order conditions are:

$$\left[\frac{\delta L}{\delta c_t}\right] = 0 \Leftrightarrow u_c(c_t, 1 - n_t^1 - n_t^2) = \lambda_t \tag{78}$$

$$\left[\frac{\delta L}{\delta n_t^1}\right] = 0 \Leftrightarrow u_l(c_t, 1 - n_t^1 - n_t^2) = \lambda_t F_n^1(k_t^1, n_t^1) \tag{79}$$

$$\left[\frac{\delta L}{\delta n_t^2}\right] = 0 \Leftrightarrow u_l(c_t, 1 - n_t^1 - n_t^2) = q_t F_n^2(k_t^2, n_t^2) \tag{80}$$

$$\left[\frac{\delta L}{\delta k_{t+1}}\right] = 0 \Leftrightarrow \beta q_{t+1}[F_n^2(k_t^2, n_t^2) - (1 - \delta)] = q_t \tag{81}$$

$$\left[\frac{\delta L}{\delta k_t^1}\right] = 0 \Leftrightarrow \lambda_t F_k^1(k_t^1, n_t^1) = q_t F_k^2(k_t^2, n_t^2) \tag{82}$$

$$\left[\frac{\delta L}{\delta \lambda_t}\right] = 0 \Leftrightarrow F^1(k_t^1, n_t^1) = c_t \tag{83}$$

$$\left[\frac{\delta L}{\delta q_t}\right] = 0 \Leftrightarrow F^2(k_t^2, n_t^2) = k_{t+1} + (1 - \delta)k_t \tag{84}$$

5 unknowns and $c_t, n_t^1, n_t^2, k_{t+1}, k_t^1$ and 5 equations every period:

(78) + (79):

$$u_l(c_t, 1 - n_t^1 - n_t^2) = u_c(c_t, 1 - n_t^1 - n_t^2) F_n^1(k_t^1, n_t^1) \tag{85}$$

(80) + (81):

$$\beta \frac{u_l(c_t, 1 - n_t^1 - n_t^2)}{F_n^1(k_{t+1}^1, n_{t+1}^1)} \left[F_n^1(k_{t+1}^1, n_{t+1}^1) - (1 - \delta)\right] = \frac{u_l(c_t, 1 - n_t^1 - n_t^2)}{F_n^1(k_t^1, n_t^1)} \tag{86}$$

(78) + (80) + (82):

$$u_c(c_t, 1 - n_t^1 - n_t^2) F_n^1(k_t^1, n_t^1) = \frac{u_l(c_t, 1 - n_t^1 - n_t^2)}{F_n^1(k_{t+1}^1, n_{t+1}^1)} F_k^2(k_t^1, n_t^1) \tag{87}$$

in addition to (83) + (84).

(7) Social Planner + Recursive formulation

Planner's problem:

$$
\begin{aligned}
V(K) &= \max_{c,i,n^1,n^2,k^1,k^2} \left\{u(c, 1-n^1-n^2) + \beta V(K')\right\} \qquad (88)\\
s.t. &: \quad c \leq F^1(k^1, n^1) \text{ (consumer goods feasibility constraint)}\\
&: \quad i \leq F^2(k^2, n^2) \text{ (capital goods feasibility constraint)}\\
&: \quad K' = (1-\delta)K + i \text{ (law of motion of capital)}\\
&: \quad n^1 + n^2 + l = 1 \text{ (time endowment)}\\
&: \quad k^1 + k^2 = K \text{ (economy wide capital)}\\
&: \quad c, K' \geq 0 \text{ (non-negativity condition)}\\
&: \quad k_0 \ > 0 \text{ given}
\end{aligned}
$$

(8) Decentralized + A.D. mkts

HH's problem:

$$
\begin{aligned}
\max_{\{c_t,i_t,n_t^{s,1},n_t^{s,2},k_t^{s,1},k_t^{s,2}\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad (89)\\
s.t. &: \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t \left[w_t(n_t^{s,1} + n_t^{s,2}) + r_t(k_t^{s,1} + k_t^{s,2}) + \pi_t^1 + \pi_t^2\right]\\
&: \quad n_t^{s,1} + n_t^{s,2} + l_t = 1\\
&: \quad x_{t+1} = (1-\delta)x_t + i_t\\
&: \quad k_t^{s,1} + k_t^{s,2} = x_t \ \forall t\\
&: \quad c_t, x_{t+1} \geq 0 \ \forall t\\
&: \quad x_0 > 0 \text{ given}
\end{aligned}
$$

Consumption goods firm's problem:

$$\max_{\{k_t^{d,1}, n_t^{d,1}\}_{t=0}^{\infty}} \pi_t^1 = F^1(k_t^{d,1}, n_t^{d,1}) - w_t n_t^{d,1} - r_t k_t^{d,1} \tag{90}$$

Capital goods firm's problem:

$$\max_{\{k_t^{d,2}, n_t^{d,2}\}_{t=0}^{\infty}} \pi^2 = F^2(k_t^{d,2}, n_t^{d,2}) - w_t n_t^{d,2} - r_t k_t^{d,2} \tag{91}$$

Note: given that factors of production are mobile across sectors, the rental rates for capital and labour will be equal **in equilibrium**. If not, more labour and capital would be supplied to the sector that paid the highest wage/rental rate eventually correcting this imbalance.

(9) Decentralized + Sequential markets[4]

Given that we have more than one sector and since technology used in each sector could be different (and hence so would the corresponding sectoral productivities), there is no reason to believe that the relative price of consumption and investment goods will be unity. As such, we can normalize the price of the investment good to 1 and define the price of the consumption good in terms of the investment one as p_t^c.

HH's problem:

$$\begin{aligned}
\max_{\{c_t,b_{t+1},i_t,n_t^{s,1},n_t^{s,2},k_t^{s,1},k_t^{s,2}\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t,l_t) \\
s.t. &: p_t^c c_t + i_t + q_t b_{t+1} \leq w_t^1 n_t^{s,1} + w_t^2 n_t^{s,2} + r_t^1 k_t^{s,1} + r_t^2 k_t^{s,2} + \pi_t^1 + \pi_t^2 + b_t \\
&: n_t^{s,1} + n_t^{s,2} + l_t = 1 \\
&: x_{t+1} = (1-\delta)x_t + i_t \\
&: k_t^{s,1} + k_t^{s,2} = x_t \ \forall t \\
&: c_t, x_{t+1} \geq 0 \ \forall t \\
&: q_t = \frac{1}{r_{t+1}} \\
&: b_{t+1} \geq -\bar{B} \\
&: x_0 > 0 \ \wedge b_0 \geq 0 \text{ given}
\end{aligned} \tag{92}$$

Consumption goods firm's problem:

$$\begin{aligned}
\max_{\{k_t^{d,1},n_t^{d,1}\}_{t=0}^{\infty}} \pi_t^1 &= p_t^c y_t^1 - w_t n_t^{d,1} - r_t k_t^{d,1} \\
s.t. &: y_t^1 = F(k_t^{d,1}, n_t^{d,1})
\end{aligned} \tag{93}$$

[4]Note that you do not really need bonds in this sequential formulation. Bonds are useful for two things: i) to pin down the interest rate in equilibrium (see appendix) and ii) to prove the equivalence between A.D. and Sequential markets.

Capital goods firm's problem:

$$\max_{\{k_t^{d,2},n_t^{d,2}\}_{t=0}^{\infty}} \pi^2 = y_t^2 - w_t n_t^{d,2} - r_t k_t^{d,2} \tag{94}$$
$$s.t. \quad : \quad y_t^2 = F(k_t^{d,2}, n_t^{d,2})$$

Performing the usual substitutions of equality constraints, the HH lagrangean would be:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1 - n_t^1 - n_t^2) + \lambda_t \left[b_t + r_t^1 k_t^1 + r_t^2 (k_t - k_t^1) + w_t^1 n_t^1 + w_t^2 n_t^2 - p_t^c c_t - q_t b_{t+1} - k_{t+1} - (1-\delta) k_t \right] \right\} \tag{95}$$

The HH's first order conditions are:

$$\left[\frac{\delta L}{\delta c_t} \right] = 0 \Leftrightarrow u_c(c_t, 1 - n_t^1 - n_t^2) = \lambda_t p_t^c \tag{96}$$

$$\left[\frac{\delta L}{\delta n_t^1} \right] = 0 \Leftrightarrow u_l(c_t, 1 - n_t^1 - n_t^2) = \lambda_t w_t^1 \tag{97}$$

$$\left[\frac{\delta L}{\delta n_t^2} \right] = 0 \Leftrightarrow u_l(c_t, 1 - n_t^1 - n_t^2) = \lambda_t w_t^2 \tag{98}$$

$$\left[\frac{\delta L}{\delta xx} \right] = 0 \Leftrightarrow ... \tag{99}$$

$$\left[\frac{\delta L}{\delta xx} \right] = 0 \Leftrightarrow ... \tag{100}$$

(10) Decentralized + Recursive formulation (RCE)

Let K represent the aggregate capital stock in the economy.

Let k represent the individual capital stock of the household.

Let the price of consumption goods in terms of investment goods be $p_c(K)$. Also, let $w(K)$ and $r(K)$ be the rental price of labour and capital in terms of the investment good. Note that all prices will be functions of the aggregate capital stock.

HH problem:

$$\begin{aligned}
V(K,k) &= \max_{c,i,n_1,n_2,k_1,k_2} \{u(c,l) + \beta V(K',k)\} && (101)\\
s.t. &: p_c(K)c + k^{'} \le w^1(K)n_1 + w^2(K)n_2 + r^1(K)k_1^d + r^2(K)k_2^d + \pi^1 + \pi^2 \text{ (bc)}\\
&: k' = (1-\delta)k + i \text{ (law of motion of capital)}\\
&: n_1 + n_2 + l = 1 \text{ (time endowment)}\\
&: k_1 + k_2 \le k \text{ (HH capital stock)}\\
&: c, k' \ge 0 \text{ (non-negativity condition)}\\
&: k_0 > 0 \text{ given}
\end{aligned}$$

Consumption goods firm's problem:

$$\begin{aligned}
\max_{k_1^d,n_1^d} \pi^1 &= p^c y^1 - w^1(K)n_1^d - r^1(K)k_1^d && (102)\\
s.t. &: y^1 = F(k_1^d, n_1^d)
\end{aligned}$$

Capital goods firm's problem:

$$\begin{aligned}
\max_{k_2^d,n_2^d} \pi^2 &= y^2 - w^2(K)n_2^d - r^2(K)k_2^d && (103)\\
s.t. &: y^2 = F(k_2^d, n_2^d)
\end{aligned}$$

Note that since the resources are mobile across sectors, the return on factors of production will be equal (in equilibrium) across them.

(11) Social planner + Sequential formulation

Planner's problem :

$$
\begin{aligned}
\max_{\{c_t, i_t^1, i_t^2, l_t\}_{t=0}^{\infty}} U \quad &= \quad \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) && (104)\\
s.t. \quad &: \quad c_t \leq F^1(k_t^1, n_t^1) \text{ (sectoral resource constraint)}\\
&: \quad i_t \leq F^2(k_t^2, n_t^2) \text{ (sectoral resource constraint)}\\
&: \quad n_t^1 + n_t^2 + l_t = 1 \text{ (time endowment)}\\
&: \quad c_t, k_{t+1}^1, k_{t+1}^2 \geq 0 \ \forall t \text{ (non-negativity condition)}\\
&: \quad k_0^1, k_0^2 > 0 \text{ given}
\end{aligned}
$$

(12) Social planner + Recursive formulation

Planner's problem:

$$
\begin{aligned}
V(K) &= \max_{c,l,i^1,i^2} \{u(c,l) + \beta V(K')\} && (105)\\
s.t. &: \; c \leq F^1(k^1, n^1) \text{ (sectoral resource constraint)}\\
&: \; i \leq F^2(k^2, n^2) \text{ (sectoral resource constraint)}\\
&: \; k^{1\prime} = (1-\delta)k^1 + i^1 \text{ (law of motion of capital)}\\
&: \; k^{2\prime} = (1-\delta)k^2 + i^2 \text{ (law of motion of capital)}\\
&: \; n^1 + n^2 + l = 1 \text{ (time endowment)}\\
&: \; c, k^{1\prime}, k^{2\prime} \geq 0 \text{ (non-negativity condition)}\\
&: \; k_0^1,\; k_0^2 > 0 \text{ given}
\end{aligned}
$$

(13) Decentralized + A.D. markets

HH problem:

$$
\begin{aligned}
\max_{\{c_t, i_t, n_t^s, k_t^s\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t, l_t) && (106)\\
s.t. &: \; \sum\nolimits_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum\nolimits_{t=0}^{\infty} p_t(w_t n_t^s + r_t k_t^s + \pi_t) \text{ (PVBC)}\\
&: \; l_t + n_t^s = 1 \text{ (time endowment)}\\
&: \; x_{t+1} = (1-\delta)x_t + i_t \text{ (law of motion of capital)}\\
&: \; k_t^s \leq x_t, \forall t\\
&: \; c_t, x_{t+1} \geq 0 \; \forall t\\
&: \; x_0 \text{ given}
\end{aligned}
$$

The associated lagrangean would be:

$$L = \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t, c_{t-1}, l_{t-1}) + \lambda[\sum_{t=0}^{\infty} p_t(w_t n_t^s + r_t k_t^s + \pi_t - c_t - k_{t+1} + (1-\delta)k_t)] \right\} \tag{107}$$

Firm's problem:

$$\begin{aligned} \max_{\{y_t, n_t^d, k_t^d\}_{t=0}^{\infty}} \pi_t &= \sum\nolimits_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t n_t^d) \\ s.t. &: \quad y_t = F(k_t^d, n_t^d) \\ &: \quad y_t, k_t^d, n_t^d \geq 0 \ \forall t \end{aligned} \tag{108}$$

Definition 7 *A competitive AD equilibrium is a sequence of prices* $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *and the HH* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t\right\}_{t=0}^{\infty}$ *such that:*

(i) Given prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t\right\}_{t=0}^{\infty}$ *solve the HH optimization's problem.*

(ii) Given prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$*, the allocations* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *solve the firm's optimization's problem.*

(iii) All markets clear:

$\hat{k}_t^s = \hat{k}_t^d$ *(capital services market)*

$\hat{n}_t^s = \hat{n}_t^d$ *(labour market)*

$\hat{y}_t = \hat{c}_t + \hat{\imath}_t$ *(goods market)*

(14) Decentralized + Sequential mkts

HH problem:

$$\max_{\{c_t, i_t, n_t^s, k_t^s, b_{t+1}\}_{t=0}^{\infty}} U = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, c_{t-1}, l_{t-1}) \tag{109}$$

$$
\begin{aligned}
s.t. \quad &: \quad c_t + i_t + q_t b_{t+1} \leq w_t n_t^s + r_t k_t^s + \pi_t + b_t \text{ (period budget constraint)} \\
&: \quad x_{t+1} = (1-\delta)x_t + i_t \text{ (law of motion of capital)} \\
&: \quad l_t + n_t^s = 1 \text{ (time endowment)} \\
&: \quad k_t^s \leq x_t \ \forall t \\
&: \quad q_t = \frac{1}{r_{t+1}} \\
&: \quad b_{t+1} \geq -\bar{B} \text{ (no Ponzi game condition)} \\
&: \quad c_t, x_{t+1} \geq 0 \ \forall t \\
&: \quad x_0 \text{ and } b_0 \text{ given}
\end{aligned}
$$

Firm's problem:

$$\max_{\left\{y_t, n_t^d, k_t^d\right\}_{t=0}^{\infty}} \pi_t = y_t - r_t k_t^d - w_t n_t^d \tag{110}$$

$$
\begin{aligned}
s.t. \quad &: \quad y_t = F(k_t^d, n_t^d) \\
&: \quad y_t, k_t^d, n_t^d \geq 0 \ \forall t
\end{aligned}
$$

The associated lagrangean would be:

$$L = \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t, 1-n_t, c_{t-1}, l_{t-1}) + \lambda_t [w_t n_t^s + r_t k_t^s + \pi_t - c_t - k_{t+1} + (1-\delta)k_t] \right\} \tag{111}$$

Definition 8 *A competitive sequential markets equilibrium is a sequence of prices* $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *and the HH* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t, \hat{b}_{t+1}\right\}_{t=0}^{\infty}$ *such that:*

(i) Given prices $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$, *the allocations* $\left\{\hat{c}_t, \hat{k}_t^s, \hat{n}_t^s, \hat{\imath}_t, \hat{b}_{t+1}\right\}_{t=0}^{\infty}$ *solve the HH optimization's problem.*

(ii) Given prices $\{\hat{q}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$, *the allocations* $\left\{\hat{y}_t, \hat{k}_t^d, \hat{n}_t^d\right\}_{t=0}^{\infty}$ *solve the firm's optimization's problem.*

(iii) All markets clear:

$\hat{k}_t^s = \hat{k}_t^d$ *(capital services market)*

$\hat{n}_t^s = \hat{n}_t^d$ *(labour market)*

$\hat{y}_t = \hat{c}_t + \hat{\imath}_t + b_{t+1}$ *(goods market)*

$\hat{b}_{t+1} = 0$ *(asset market)*

(15) Decentralized + Recursive formulation (RCE)

Let K represent the aggregate capital stock in the economy.

Let N represent the aggregate hours worked in the economy.

Let C represent the aggregate level of private consumption in the economy.

Let k represent the individual capital stock of the household.

Let n represent the individual hours worked by the household.

Let c represent the individual household consumption of the numeraire good.

Let $r(K, C_{-1}, N_{-1})$ and $w(K, C_{-1}, N_{-1})$ be the rental prices for capital and labor in the economy in terms of the consumption good.

HH problem:

$$
\begin{aligned}
V(K, C_{-1}, N_{-1}, k, c_{-1}, l_{-1}) &= \max_{c,i,l} \{u(c, l, c_{-1}, l_{-1}) + \beta V(K', k', C, N, c, l)\} \qquad (112) \\
s.t. &: \quad c + i \leq r(K)k + w(K)n + \pi \\
&: \quad k' = (1-\delta)k + i \text{ (law of motion for individual capital)} \\
&: \quad l + n = 1 \text{ (time constraint)} \\
&: \quad c, k' \geq 0 \ \forall t \geq 0 \text{ (non-negativity)} \\
&: \quad (K', C, N) = G(K, C_{-1}, N_{-1}) \text{ (perceived LOM of the Agg state)}
\end{aligned}
$$

Firm's problem:

$$\max_{k^d, n^d} \pi = F(k^d, n^d) - w(K)n^d - r(K)k^d \qquad (113)$$

Appendix:

This appendix contains notes on various topics that have come up on discussions and seemed useful to keep as references. They are not organized under any particular logic or order.

1 - About the rational expectations equilibrium assumption:

In a sequence economy, present actions have future consequences. As such, to make a reasonable decision in the present, agents must make the effort to envision what the future has in store. This effort is costly and time and energy must be expended to acquire updated information about the current environment, mental resources must be harnessed to envision possible future scenarios. There are a few different approaches to modeling the forecasts of economic agents.

1. Naive expectations: agents expect tomorrow to be just like today.

2. Adaptive expectations: agents use some mechanical function of past events to predict future ones. Some more refined versions of adaptive expectations propose an evolutionary approach, where people keep their perdition rules until they stop working and then replace them for updated ones.

3. Rational expectations: agents know the true probability distribution of all future events. An implication is that they know exactly how the economy works.

4. Perfect Foresight: agents know exactly what will happen in the future. It can be seen as a special case of rational expectations where the probability distribution of future events is degenerate (all events have probability one or zero).

Note that under a rational expectation assumption, agents in the economy are able to fully understand their economic context and adequately forecasts the evolution of macroeconomic aggregates. As such, as we set up the RCE we must include their beliefs about the law of motion of the aggregate states. In turn, these beliefs will be true in equilibrium and agents will avoid making systematic mistakes. This is in contrast to, for example, an adaptive expectations context where agents will update their forecasts based on the errors made in previous periods. In such a set up, forecast errors will be serially correlated and agents would thus tend to make systematic mistakes.

2 - About the Transversality Condition (TVC):

The transversality condition (TVC) for an infinite horizon dynamic optimization problem is one of the boundary conditions required to solve the problem (in addition to the problem's first-order and initial conditions). The transversality condition requires the present value of the state variables to converge to zero as the planning horizon recedes towards infinity. Intuitively, we can think that the first order conditions determine what is optimal from a period to period perspective, but might ignore the "big picture". As such, in an infinite horizon problem the equivalent of a terminal condition is needed. The TVC is such condition. In a sequential set-up the TVC is usually expressed as:

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0 \tag{114}$$

This states that the value of the capital stock (in this case the only state variable), when measured in terms of discounted utility, should equal 0. In this particular example $f'(k_t)$ represents the price of capital goods at every period. Hence the condition requires that the present value of the capital stock in period t, evaluated using period t market prices tends to zero as t tends to infinity. Note that the condition only requires that the (shadow) value of the capital stock (and not the capital stock itself) has to converge to zero. In turn, this can also be expressed as:

$$\lim_{t\to\infty} \beta^t u' \left(F(k_t, 1) - (1-\delta)k_t - k_{t+1}\right) f'(k_t) k_t = 0$$

Let $f(k_t) = F(k_t, 1) - (1-\delta)k_t$

$$\lim_{t\to\infty} \beta^t u' \left[f(k_t) - k_{t+1}\right] f'(k_t) k_t = 0 \tag{115}$$

Remark: An alternative formulation of the TVC is:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{116}$$

where λ_t is the lagrange multiplier on the constraint $c_t + k_{t+1} = f(k_t)$ and from the FOC we know it is equal to the discounted marginal utility of consumption at time t:

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t \\ \beta^t u'(f(k_t) - k_{t+1}) &= \lambda_t \end{aligned} \tag{117}$$

Replacing the above into (116) yields:

$$\lim_{t\to\infty} \beta^t u'(f(k_t) - k_{t+1}) k_{t+1} = 0 \tag{118}$$

which is equivalent to:

$$\lim_{t\to\infty} \beta^{t-1} u'(f(k_{t-1}) - k_t) k_t = 0 \tag{119}$$

Moreover, from the Euler equation we have that:

$$u'(c_{t-1}) = \beta u'(c_t) f'(k_t)$$

Replace into (120) to get:

$$\begin{aligned} \lim_{t\to\infty} \beta^{t-1} \beta u'(c_t) f'(k_t) k_t &= 0 \\ \lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t &= 0 \end{aligned} \tag{120}$$

which is equivalent to (114).

Moreover, in an AD market structure the TVC becomes:

$$\lim_{t\to\infty} p_t \lambda k_{t+1} = 0 \tag{121}$$

where p_t is the typical A.D. price.

The first-order and transversality conditions are jointly sufficient to identify an optimum in a concave optimization problem (see Stockey & Lucas p98-99). Hence, given an optimal path, the necessity of the transversality condition reflects the impossibility of finding an alternative feasible path for which each state variable deviates from the optimum at each time and increases discounted utility. However, is the TVC necessary? The answer to this question turns out to be quite complex. For log utility and a neoclassical utility function Ekelund and Scheikman (1985) have shown that the TVC is in fact a necessary condition (see Krueger notes p.47).

It is very important to be aware that the transversality condition is not something that needs to be imposed so that a solution to the optimization problem exists. It is not the same as a no Ponzi scheme condition. That is why the no Ponzi condition is a constraint in the maximization problem, while the TVC is a condition is one of the optimality conditions that needs to be satisfied by every solution. In other words, the TVC will restrict the paths that actually satisfy the Euler equation to find those (or that) which comprise a solution to the optimization problem. Last, note that the TVC is only required when the optimization problem is of an infinite horizon. (i.e.: non-infinite or recursive formulations do not require a TVC).

Example 1:

Consider the deterministic neoclassical growth model. The Euler equation and the capital accumulation equation are:

$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1}) f'(k_{t+1}) \\
k_{t+1} &= f(k_t) - c_t
\end{aligned}
$$

where $F(K_t, N_t)$ is CRS and $f(k_t) = F(K, 1) + (1-\delta)k_t$. Thus we have a system of two equations and two unknowns: c_t and k_{t+1} which we can solve. Iterating forward yields:

$$c_{t+1} = f(k_{t+1}) - k_{t+2}$$

Replacing the above into the Euler equation, yields the following second order difference equation.

$$\beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = u'(f(k_t) - k_{t+1})$$

In order to have a unique solution, a second-order difference equation requires to be supplemented with exactly two boundary conditions. The first boundary condition is the initial value of capital k_0. Note that if we would know the value of c_0 our problem would be solved. Unfortunately we do not, and therefore need to provide an alternative second boundary condition. The TVC will play that role. It will turn out that there will be only one value of c_0 that doesn't violate the TVC.

Example 2:

Consider the same model as before but in its decentralized version and where, in addition to capital, individuals may also save in terms of one period risk-less bonds. The problem set-up is:

$$\max_{\{c_t,k_{t+1},b_{t+1}\}_{t=0}} U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad : \quad r_t k_t + b_t \geq c_t + k_{t+1} - (1-\delta)k_t + \frac{b_{t+1}}{R_t}$$

Assuming that $F(K_t, N_t) = AK_t^\alpha N_t^{1-\alpha}$ where $A > 0$ is some constant known value, the equations that characterize the equilibrium are: [5]

$$\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1}) R_t \\
u'(c_t) &= \beta u'(c_{t+1})[r_{t+1} + (1-\delta)] \\
r_t k_t + b_t &= c_t + k_{t+1} - (1-\delta)k_t + \frac{b_{t+1}}{R_t} \\
r_t &= \alpha A k_t^{\alpha-1} \\
R_t &= \alpha A k_t^{\alpha-1} + 1 - \delta
\end{aligned}$$

Note that because there is no uncertainty and resources may be free be invested in either bonds or physical capital, both assets will yield equal returns in equilibrium[6].

[5] From the FOC of the firm: $r_t = \alpha A K_t^{\alpha-1} N_t^{1-\alpha} = \alpha A \left(\frac{K_t}{N_t}\right)^\alpha \frac{N_t}{K_t}$

[6] In a context with uncertainty this would be different, as there would be a risk premium consideration. If A_t is stochastic, the FOCs would then be:

$$\begin{aligned}
u'(c_t) &= \beta R_t E_t[u'(c_{t+1})] \\
u'(c_t) &= \beta E_t[u'(c_{t+1})(r_{t+1} + 1 - \delta)] \\
&= \beta[E_t u'(c_{t+1})(E_t(r_{t+1}) + 1 - \delta) + Cov(u'(c_{t+1}), r_{t+1})] \\
r_t k_t + b_t &= c_t + k_{t+1} - (1-\delta)k_t + \frac{b_{t+1}}{R_t} \\
r_t &= \alpha A_t k_t^{\alpha-1}
\end{aligned}$$

$$\Rightarrow \quad \beta R_t E_t[u'(c_{t+1})] = \beta[E_t u'(c_{t+1})(E_t(r_{t+1}) + 1 - \delta) + Cov(u'(c_{t+1}), r_{t+1})]$$

$$\Rightarrow \quad R_t = E_t(r_{t+1}) + 1 - \delta + \frac{Cov(u'(c_{t+1}), r_{t+1})]}{E_t[u'(c_{t+1})]}$$

Substituting consumption out yields a system of second-order difference equations:

$$\begin{aligned}
u'\left(r_t k_t + b_t - k_{t+1} + (1-\delta)k_t - \frac{b_{t+1}}{R_t}\right) &= \beta u'\left(r_{t+1}k_{t+1} + b_{t+1} - k_{t+2} + (1-\delta)k_{t+1} - \frac{b_{t+2}}{R_{t+1}}\right) R_t \\
u'\left(r_t k_t + b_t - k_{t+1} + (1-\delta)k_t - \frac{b_{t+1}}{R_t}\right) &= \beta u'\left(r_{t+1}k_{t+1} + b_{t+1} - k_{t+2} + (1-\delta)k_{t+1} - \frac{b_{t+2}}{R_{t+1}}\right) [r_{t+1} + (1-\delta)]
\end{aligned}$$

Further substitution yields:

$$u'\left(\alpha A k_t^{\alpha-1} k_t + b_t - k_{t+1} + (1-\delta)k_t - \frac{b_{t+1}}{R_t}\right) = \beta u'\left(\alpha A k_{t+1}^{\alpha-1} k_{t+1} + b_{t+1} - k_{t+2} + (1-\delta)k_{t+1} - \frac{b_{t+2}}{R_{t+1}}\right) R_t$$

$$u'\left(\alpha A k_t^{\alpha-1} k_t + b_t - k_{t+1} + (1-\delta)k_t - \frac{b_{t+1}}{R_t}\right) =$$
$$\beta u'\left(\alpha A k_{t+1}^{\alpha-1} k_{t+1} + b_{t+1} - k_{t+2} + (1-\delta)k_{t+1} - \frac{b_{t+2}}{R_{t+1}}\right) [\alpha A k_{t+1}^{\alpha-1} + (1-\delta)]$$

Since, in equilibrium, $R_t = r_t + 1 - \delta$, the system above boils down to a single equation:

$$u'\left(\alpha A k_t^{\alpha-1} k_t + b_t - k_{t+1} + (1-\delta)k_t - \frac{b_{t+1}}{\alpha A k_t^{\alpha-1} + (1-\delta)}\right) =$$
$$\beta u'\left(\alpha A k_{t+1}^{\alpha-1} k_{t+1} + b_{t+1} - k_{t+2} + (1-\delta)k_{t+1} - \frac{b_{t+2}}{\alpha A k_{t+1}^{\alpha-1} + (1-\delta)}\right) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

Arguably this is not a simple expression, but this equation has only (k_t, k_{t+1}, k_{t+2}) and (b_t, b_{t+1}, b_{t+2}) as arguments. All the remaining elements are the parameters $(\alpha, \beta, A, \delta)$ and of course the derivative of the utility function which still needs to be specified (and might add a few more parameters). Yet, mathematically speaking, the expression above has become a system of second-order difference equations (there are no more prices anymore) and for it to have a unique solution, the system requires four boundary conditions. The initial stock of capital k_0 and bonds b_0 would provide two of those. The remaining two constraints would fall on the following transversality conditions:

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0$$
$$\lim_{t\to\infty} \beta^t u'(c_t) b_t = 0$$

Note that earlier we mentioned that the TVC required the present value of the state variable evaluated using period t prices to converge to zero. In the case of the capital stock, the relevant price was $f'(k_t)$. In the case of the risk-free bond that delivers one unit of the consumption good every period, its price is in effect 1.

3 - About the no Ponzi scheme condition:

In a sequential formulation where agents have access to financial instruments (bonds), lending and borrowing can be used to transfer wealth between periods. This could, in principle, lead to a Ponzi Scheme understood as a fraudulent investment operations where the operator (an individual or firm) pays returns to investors from new capital rather from profits earned by successful investments. As such, in an infinite horizon problem an agent could maximize his/her lifetime utility simply by rolling over its debt position indefinitely.

In order to prevent Ponzi schemes, we shall need to impose some restrictions on asset trades. However we would like to impose the weakest possible limits so that it prevents Ponzi schemes, but the constraint does not really bind in equilibrium. In other words, in the optimal solution, we want a borrowing limit that will not be binding.

This debt limit is usually referred to as the “natural” debt limit and it comes from the commonsense requirement that is has to be feasible for the consumer to repay this debt in every possible state (should there be more than one state). In the case of an endowment economy this could be the discounted stream of all future endowments; while in the case of a production economy the natural borrowing limit could be the discounted stream of all future income payments. For example, in case of future labor income payments:

$$\begin{aligned} b_{t+1} &\geq -\bar{B} \\ &: \quad \bar{B} \equiv \sum_{j=i}^{\infty} \frac{w_{t+j} n_{t+j}}{R_{t+1} \cdots R_{t+j}} \end{aligned}$$

4 - About the number of constraints and lagrange multipliers:

It should be noted that we always need one lagrange multiplier per constraint. Consider for example a two period optimization problem:

$$\begin{aligned} & \max u(c_0) + \beta u(c_1) \\ s.t. \quad &: \quad y_0 \geq c_0 \\ &: \quad y_1 \geq c_1 \end{aligned} \tag{123}$$

which yields the following objective function:

$$L(c_0, c_1, \lambda_0, \lambda_1) = u(c_0) + \beta u(c_1) + \lambda_0 (y_0 - c_0) + \lambda_1 (y_1 - c_1) \tag{124}$$

If we would have an infinite amount of periods and constraints we can write:

$$\begin{aligned} L &= \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t) + \sum\nolimits_{t=0}^{\infty} \lambda_t (y_t - c_t) \\ &= \sum\nolimits_{t=0}^{\infty} \left\{ \beta^t u(c_t) + \lambda_t (y_t - c_t) \right\} \end{aligned} \tag{125}$$

5 - About setting up the lagrangean and the meaning of the multiplier:

Consider a typical lagrangean for a social planner formulation:

$$L=\sum_{t=0}^{\infty}\left\{\beta^t u(c_t,1-n_t)+\lambda_t[F(k_t,n_t)-c_t-k_{t+1}+(1-\delta)k_t]\right\}$$

Note this could also be written as:

$$L=\sum_{t=0}^{\infty}\beta^t\left\{u(c_t,1-n_t)+\lambda_t[F(k_t,n_t)-c_t-k_{t+1}+(1-\delta)k_t]\right\}$$

These two forms are equivalent. Their only differ in the interpretation of λ_t. The first formulation is usually referred to as the present value lagrangean, while the second one as the current value multiplier.

6 - About taking FOC's under uncertainty:

Consider the following optimization problem under uncertainty:

$$\begin{aligned}
\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} U &= E_0\sum_{t=0}^{\infty}\beta^t u(c_t)\\
s.t. &: c_t+k_{t+1}\leq e^{z_t}F(k_t,n_t)+(1-\delta)k_t \text{ (resource/feasibility constraint)}\\
&: n_t+l_t=1 \text{ (time endowment)}\\
&: z_{t+1}=\rho z_t+\varepsilon_{t+1} \text{ (law of motion of the shock)}\\
&: \lim_{t\to\infty}E_t[\lambda_t k_{t+1}]=0 \text{ (transversality condition)}\\
&: c_t,k_{t+1}\geq 0\ \forall t \text{ (non-negativity condition)}\\
&: z_0,k_0>0 \text{ given}
\end{aligned}$$

The lagrangean for this problem would be:

$$L=E_0\sum_{t=0}^{\infty}\beta^t\left\{u(c_t)+\lambda_t[e^{z_t}F(k_t,n_t)+(1-\delta)k_t-c_t-k_{t+1}]\right\} \tag{126}$$

FOCs:

$$
\begin{aligned}
[c_t] \quad &: \quad \beta^t u'(c_t) = \lambda_t \quad \forall t \wedge \forall s^t \in S^t \\
[k_{t+1}] \quad &: \quad E_t[\lambda_{t+1}(e^{z_t} F_k'(k_{t+1}, n_{t+1}) + (1-\delta))] = \lambda_t \quad \forall t \wedge \forall s^t \in S^t
\end{aligned}
$$

Note that the original E_0 was replaced by E_t: the agent's expectation conditional on his information set at time t. This happens because the FOCs must hold for every t and every $s^t \in S^t$.

7 - About introducing shocks into models:

The main way in which we have introduced uncertainty has been through simple TFP shocks. Depending on how the shock is introduced into the production function we'll have a different laws of motion for it. In the case of a TFP shock in particular we would like to avoid negative values. A few examples:

$$
\begin{aligned}
\text{Production Function} \quad &: \quad e^{z_t} F(k_t, n_t) \\
\text{Shock's law of motion} \quad &: \quad z_{t+1} = \rho z_t + \varepsilon_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\text{Production Function:} \quad & \quad z_t F(k_t, n_t) \\
\text{Shock's law of motion (1)} \quad &: \quad z_{t+1} = z_t^{\rho} e^{\varepsilon_{t+1}} \\
\text{Shock's law of motion (2):} \quad & \quad \ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1} \\
\text{Shock's law of motion (3)} \quad &: \quad \ln z_{t+1} = (1-\rho)\bar{z} + \rho \ln z_t + \varepsilon_{t+1}
\end{aligned}
$$

In all cases $\{\varepsilon_t\}$ is assumed to be a sequence of *iid* (usually normally distributed) random variables drawn in every period from the same distribution with zero mean and variance σ_z^2. Note that the only difference between the last two laws of motion is that the unconditional mean for $\{z_t\}$ would be 0 in the first case and $\bar{z}$ in the second one.

Last uncertainty could also be added to the representative agent's utility function. In that case, rather than a productivity shock, we are introducing a preference or taste shock.

8 - About the properties of the utility function:

In our models the representative agent will have a utility function such that: $u(c_t) : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ (note that we allow for negative utility since it is only a cardinal measure). Below the main properties we often require utility functions to have in our models:

a) *Time Separability*

Agents in the economy seek to maximize the discounted sum of period (or instantaneous) utility. Usually, the period utility at time t depends on some argument solely in period t (eg: consumption, leisure). If, however, the agent experiences habit persistance, then each period utility could also depend on past consumption.

b) *Time Discounting*

The fact that $\beta < 1$ indicates that agents are impatient. As such, the same amount of consumption yields less utility if it comes at a later time in an agent's life.

$\beta = \frac{1}{1+\rho}$, where β is the discount factor and ρ is the discount rate.

c) *Differentiability and continuity*

The period utility function should be:

i - continuous and bounded

ii - continuously differentiable

iii - strictly increasing: $u'(c) > 0$

iv - strictly concave: $u''(c) < 0$

v - satisfies inada conditions:

$\lim_{c\rightarrow 0} u'(c) = \infty$

$\lim_{c\rightarrow \infty} u'(c) = 0$

Note that:

v + $k_0 > 0$: rules out corner solutions

iv: implies risk aversion and diminishing marginal utility of consumption

i + iii + iv: guarantee a unique solution

9 - About the properties of the production function:

A production function describes the technological relationship between quantities of inputs, such as capital, land and labor; and the quantity of output that they allow to produce:

$$Y_t = z_t F(K_t, N_t, L_t)$$

where K_t, N_t and L_t are capital, labor and land respectively. The term z_t is usually referred to as total factor productivity. It captures variations in output not explained by variations in inputs. When z_t increase, output rises even though inputs might stay the same. This z_t is often seen as a measure of the productivity of inputs. There are (at least) four standard assumptions usually made on the neoclassical production function:

1. The production function is increasing in both inputs:

$$F_n(K, N) > 0$$
$$F_k(K, N) > 0$$

   hence the marginal product of each input is strictly positive.

2. The production function exhibits decreasing marginal returns in each input: keeping capital (labor) fix, the marginal product of labor (capital) decreases as the quantity of labor (capital)

used in production increases. Alternatively:

$$F_{nn}(K, N) < 0$$

$$F_{kk}(K, N) < 0$$

3. The production function exhibits constant returns to scale. In fought terms, doubling the inputs doubles the output. More rigorously:

$$z_t F(\lambda K_t, \lambda N_t) = \lambda z_t F(K_t, N_t) = \lambda Y_t$$

for any constant $\lambda > 1$.

4. Essentiality of both factors of production:

$$z_t F(K_t, 0) = 0$$

$$z_t F(0, N_t) = 0$$

10 - About the Negishi method to compute equilibria:

The idea of this method is to:

1) Find all the PO allocations.

2) Identify the CE from the set of PO allocations.

The Social Planner's (SP) problem (follows from the endowment economy described at the beginning of these notes):

$$
\begin{aligned}
& \max_{\{(c_t^1, c_t^2)\}_{t=0}^{\infty}} \alpha u(c_t^1) + (1-\alpha) u(c_t^2) && (127) \\
s.t. \quad : \quad & c_t^1 + c_t^2 \leq e_t^1 + e_t^2 \quad \forall t \ \ \text{(resource constraint)} && (128) \\
: \quad & c_t^i \geq 0 \ \ \forall t, \ \ \forall i = 1, 2 \ \ \text{(non-negativity constraint)} && (129)
\end{aligned}
$$

for a Pareto weight $\alpha \in [0, 1]$. Intuitively, this weight indicates how important agent's 1 utility is to the planner, relative to agent's 2 utility. Note that the Social Planner replaces the market mechanism and that is why there are no prices in this optimization problem. The Social Planner is an alternative, centralized allocation mechanism to the market system.

Naturally, the solution to this problem will be a function of the Pareto weights (ie: the optimal consumption choice will be a function of α).

Assuming log utility, the SP objective function becomes:

$$
L(c_t^1, c_t^2, \mu_t) = {}_{t=0}^{\infty} \left\{ \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha) \ln(c_t^2) \right] + \frac{\mu}{2} [e_t^1 + e_t^2 - c_t^1 - c_t^2] \right\} \tag{130}
$$

FOCs:

$$
\begin{aligned}
[c_t^1] \quad : \quad & \frac{\alpha \beta^t}{\tilde{c}_t^1} = \frac{\mu_t}{2} && (131) \\
[c_t^2] \quad : \quad & \frac{(1-\alpha)\beta^t}{\tilde{c}_t^2} = \frac{\mu_t}{2} && (132)
\end{aligned}
$$

Combining yields:

$$
\begin{aligned}
\frac{\tilde{c}_t^1}{\tilde{c}_t^2} \quad &= \quad \frac{\alpha}{1-\alpha} && (133) \\
&\Rightarrow \quad \tilde{c}_t^1 = \frac{\alpha}{1-\alpha} \tilde{c}_t^2 && (134)
\end{aligned}
$$

A higher Pareto weight for agent 1 results in this agent receiving more consumption in every period relative to agent 2.

From the resource constraint we know that:

$$\tilde{c}_t^1 + \tilde{c}_t^2 = 2 \quad \forall t \tag{135}$$

Combining with (133) yields:

$$\begin{aligned} \frac{\alpha}{1-\alpha}\tilde{c}_t^2 + \tilde{c}_t^2 &= 2 \\ &\Rightarrow \tilde{c}_t^2 = 2(1-\alpha) \quad \forall t && (136) \\ &\Rightarrow \tilde{c}_t^1 = 2\alpha \quad \forall t && (137) \end{aligned}$$

The SP divides the total resources in every period according to the weights. In turn, the lagrange multipliers are given by (from the FOCs):

$$\mu_t = \frac{2\alpha\beta^t}{\tilde{c}_t^1} = \beta^t \tag{138}$$

Hence, in this economy, the set of PO allocations is given by:

$$PO = \left\{ \{(\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty} : c_t^1 = 2\alpha \wedge c_t^2 = 2(1-\alpha) \text{ for some } \alpha \in [0,1] \right\}$$

How is this helpful in finding the competitive equilibrium for this economy? We compare FOCs:

(a) SP FOC for agent 1:

$$\frac{\beta^t}{\tilde{c}_t^1} = \frac{\mu_t}{2\alpha}$$

(b) CE FOC for agent 1:α

$$\frac{\beta^t}{\hat{c}_t^1} = \lambda_1 p_t$$

By picking $\lambda_1 = \frac{1}{2\alpha}$ and $p_t = \beta^t$ these first order conditions are identical. Similarly, if we pick $\lambda_2 = \frac{1}{2(1-\alpha)}$ the same is true for agent 2. This means that for the appropriate choices of individual Lagrange multipliers λ_i and prices p_t, the optimality conditions for the SP problem and for the household optimization problem coincide.

Given that we have found an unique CE but many PO allocations (depending on the α), there is an additional requirement that a CE imposes which the planner does not require. This is the problem of feasibility. The SP deals with the economy's resource constraint, while in a CE each household is constrained by their budgets. As such, the last step to single out CE equilibrium allocations from the set of PO allocations is to ask which PO allocation could be affordable for all households if these were to surface as market prices the lagrange multipliers from the SP problem.

Define the transfer function $t^i(\alpha), i = 1, 2$ by:

$$t^i(\alpha) = \sum\nolimits_{t=0} [c_t^i(\alpha) - e_t^i] \tag{139}$$

The number $t^i(\alpha)$ is the amount of the numeraire good that agent i would need as trasnfer in order to be able to afford the PO allocation indexed by α. The transfer functions for agent one and two would be:

$$\begin{aligned}
t^1(\alpha) &= \sum_{t=0}^{\infty} \mu_t [c_t^1(\alpha) - e_t^1] \\
&= \sum_{t=0}^{\infty} \beta^t [c_t^1(\alpha) - e_t^1] \\
&= \frac{1}{1-\beta}\left[2\alpha - \frac{2}{1+\beta}\right] \\
& \quad \text{(since } \frac{2}{1+\beta} \text{ is how much agent 1 consumed in the CE)} \\
&= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} && (140) \\
t^2(\alpha) &= \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2} && (141)
\end{aligned}$$

To find the competitive equilibrium from the set of PO allocations we need to obtain the weight α

such that $t^1(\alpha) = t^2(\alpha) = 0$. That is to say, the PO allocations that both agents can afford with no transfers. In turn, this yields:

$$\begin{aligned} 0 &= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} \\ &\Rightarrow \quad \alpha = \frac{1}{1+\beta} \end{aligned} \tag{142}$$

resulting in the competitive allocations:

$$c_t^1 = \frac{2}{1+\beta} \tag{143}$$

$$c_t^2 = \frac{2\beta}{1+\beta} \tag{144}$$

Equilibrium prices are given by the lagrange multipliers: $\mu_t = \beta^t$

11 - About Risk and Uncertainty:

We briefly introduce here the notion of risk aversion, which is paramount in economies with uncertainty. Following MWG, we denote amounts of money by the continuous variable x and describe a monetary lottery by means of a CDF $F : \mathbb{R} \to [0, 1]$. That is, for any x, $F(x)$ is the probability that the realized payoff is less than or equal to x. With this in mind, we define a few useful concepts.

Definition 9 *A decision maker is a risk avert (or exhibits risk aversion) if for any lottery $F(.)$, the lottery that offers the expected amount $\int x dF(x)$ with certainty (i.e.: a degenerate lottery) is at least as good as the lottery $F(.)$ itself.*

If the agent's preferences admit an expected utility representation with Bernoulli utility function

$u(x)$, it follows directly from the definition above that the decision maker is risk averse if and only if:

$$\underbrace{\int u(x)dF(x)}_{\text{Expected utility of playing lottery}} \quad \leq \quad \underbrace{u\left(\int xDF(x)\right)}_{\text{Utility of lottery's expected payoff}} \qquad \forall F(.)$$

This is known as Jensen's inequality, and it is the defining property of a concave function. Hence, in the context of expected utility it is common to associate *risk aversion* with concavity of $u(.)$ and strict risk aversion with strict concavity. Strict concavity implies that the marginal utility of money is decreasing. In turn at any level of wealth, the utility gain from an extra dollar is smaller than (in absolute value) the utility loss of having a dollar less. It follows that a risk of gaining or losing a dollar with even probability is not worth taking.

The concavity of utility functions is related to the convexity of indifference curves. In standard microeconomic theory this feature of preferences represent a "taste for diversity". Under uncertainty, it represents a desire to smooth consumption across states of the world. Lastly, two useful concepts for the analysis of risk aversion:

Definition 10 *The Certainty Equivalent of $F(.)$, denoted $\bar{x}$, is the amount of money for which the individual is indifferent between taking the gamble $F(.)$ and receiving the certain amount $\bar{x}$. Formally:*

$$\underbrace{u(\bar{x})}_{\text{Utility of certainty payment}} \quad = \quad \underbrace{\int u(x)dF(x)}_{\text{Expected utility of playing lottery}}$$

Definition 11 *The Risk Premium π is the amount the individual is willing to give up (or pay) to avoid taking the gamble.*

Consider an individual with initial wealth w, who faces a lottery that pays a random amount z

with $E(z) = 0$ and $Var(z) = \sigma_z$. The premium π solves:

$$\underbrace{u(w-\pi)}_{\text{Utility of Wealth net of premium}} = \underbrace{\int u(w+z)dF(z)}_{\text{Expected utility of playing lottery}} = E[u(w+z)]$$

Now that we know what it means to be risk averse, we can try to measure the extent of risk aversion.

Measuring Risk Aversion

Definition 12 *Given a (twice differentiable) Bernoulli utility function $u(.)$ for money, the Arrow-Pratt coefficient of Absolute Risk Aversion at x is defined as:*

$$R_A(x) = -\frac{u''(x)}{u'(x)}$$

where x measures the agent's wealth level.

The intuition is as follows: we know that risk neutrality is equivalent to the linearity of $u(.)$, that is $u''(.) = 0$ for all x. Therefore it seems logical to argue that the degree of risk aversion is related to the curvature of $u(.)$. The most natural measure of the curvature of the utility function at x would be $u''(x)$. However, this is not an adequate measure because it is not invariant to positive linear transformations of the utility function. Hence, to make it invariant, the simplest modification is to divide by $u'(x)$. If we change sign so as to have a positive number for an increasing concave $u(.)$, we obtain the mention measure.

Absolute Risk Aversion is a measure of the absolute amount of wealth an individual is willing to expose to risk as a function of changes in wealth. Constant absolute risk aversion (CARA) means that the amount of wealth one is willing to expose to risk remains unchanged as wealth increases or decreases. Decreasing absolute risk aversion (DARA) means that the amount of wealth someone is willing to expose to risk increases as wealth increases, while increasing absolute risk aversion (IARA) means that one's tolerance for absolute risk exposure falls as wealth increases.

For example, consider an agent who needs to allocate an initial wealth of \$100 between a safe and risky asset. Further assume that in equilibrium he allocates \$10 on the risky asset and \$90 on the safe one. Should his initial wealth increase to \$200, CARA preferences would determine than in equilibrium he still allocates \$10 to the risky asset and the rest to the safe one. Note that it is the nominal level (price times quantity) what matters in CARA. In a general equilibrium set up, the quantity could go up, but the price could go down, maintaining constant the level of wealth invested in a particular asset class.

Definition 13 *Given a (twice differentiable) Bernoulli utility function $u(.)$ for money, the Arrow-Pratt coefficient of Relative Risk Aversion at x is defined as:*

$$R_R(x) = -x\frac{u''(x)}{u'(x)}$$

Relative Risk Aversion is a measure of one's willingness to accept risk as a function of the percentage of one's wealth that is exposed to risk. Constant relative risk aversion (CRRA) implies that the percentage of wealth one is willing to expose to risk remains unchanged as wealth increases. Decreasing relative risk aversion (DRRA) indicates that the percentage of wealth one is willing to expose to risk increases with wealth, while increasing relative risk aversion (IRRA) means that the percentage of wealth one is willing to expose to risk falls as wealth increases.

Following the previous example, an agent with CRRA preferences and an initial wealth of \$100 who allocates 10 % into risky assets in equilibrium would continue to allocate the same percentage into that asset class even after an increase in initial wealth to \$ 200. He would allocate \$ 20, more than initially but still 10 % of total wealth.

Interpreting the measures of Risk Aversion

To interpret the two measures of risk aversion it is helpful to compare a situation of risk-less wealth versus a risky gamble[7]. In this context, Arrow measured the increase in odds over a fair gamble (i.e.: one with $p = \frac{1}{2}$), that would induce the risk-averse agent into accepting the bet. In other words, how much higher than $\frac{1}{2}$ would p have to be to get the agent to accept the bet in the first place. For example if we think of a bet that pays z with probability p and $-z$ with probability $1-p$, it is evident that a risk averse agent will never accept this bet if $p = \frac{1}{2}$. The risk aversion coefficient can help us discern how much higher than $\frac{1}{2}$ does p need to be to induce the investor to accept the bet.

Similarly, Pratt sought to understand what payment would a risk averse agent be willing to make in order to avoid taking a fair gamble (i.e.: where $p = \frac{1}{2}$). Alternatively, how much wealth is an individual ready to give up in order to avoid taking risks. Both interpretations can be accommodated by both types of risk aversion coefficients. For illustration purposes I will develop the second intuition for an absolute type of bet (hence rendering the absolute risk aversion coefficient).

Consider an individual with initial wealth w_0 who faces a bet that plays a random amount z with $E(z) = 0$ and $Var(z) = \sigma_z$. What is the premium π that the individual would be willing to pay to avoid this bet? The premium π solves:

$$u(w_0 - \pi) = E[u(w_0 + z)]$$

Take a first-order Taylor expansion on the left-hand side around $\pi^* = 0$ and a second-order expan-

[7]In all these cases it will be helpful to remember the theorem stated by Brook Taylor regarding the approximation of a function f. The first and second order approximations are defined as:
f(x+a) $\approx f(x) + f'(x)a$
$f(x+a) \approx f(x) + f'(x)a + f''(x)a^2$

sion on the right-hand side around $z^* = 0$.

$$
\begin{aligned}
u(w_0 - \pi) &\approx u(w_0 - \pi^*) - u'(w_0 - \pi^*)(\pi - \pi^*) \\
&= u(w_0) - u'(w_0)\pi \\
E[u(w_0 + z)] &\approx E\left[u(w_0 + z^*) + u'(w_0 + z^*)(z - z^*) + \frac{1}{2}u''(w_0 - z^*)(z - z^*)^2\right] \\
&= u(w_0) + \frac{1}{2}u''(w_0)E(z^2) \\
&= u(w_0) + \frac{1}{2}u''(w_0)\sigma_z
\end{aligned}
$$

Equating both sides, we obtain:

$$\pi^* = -\frac{u''(w_0)}{u'(w_0)}\frac{\sigma_z}{2}$$

The premium the individual is willing to pay is proportional to the lottery's variance. The term $\left[-\frac{u''(x)}{u'(x)}\right]$ is the coefficient of proportionality and is called coefficient of absolute risk aversion. This happens because the winnings on the bet are on levels or absolute terms $(+z, -z)$. If however, we write down the bet's payoffs as a proportion of the agent's initial wealth $(+zw, -zw)$, we would obtain:

$$\pi^* = -w_0\frac{u''(w_0)}{u'(w_0)}\frac{\sigma_z}{2}$$

where the coefficient of proportionality equals the relative risk aversion parameter which measures the same type of aversion to risk, but where the bet is expressed as a percentage of wealth (or potentially income or consumption). Note also that $w_0 - \pi^*$ defines the certainty equivalent wealth for $w_0 + z$.

Last, in the case of Arrow's measure one can show that the boost in probabilities above 1/2 would be approximately equal to:

$$\begin{aligned} p &\approx \frac{1}{2} - \frac{1}{4}\left[\frac{u''(w_0)}{u'(w_0)}\right] z \\ &= \frac{1}{2} - \frac{1}{4}\, R_A(w_0) z \end{aligned}$$

$$\begin{aligned} p &\approx \frac{1}{2} - \frac{1}{4}\left[w_0 \frac{u''(w_0)}{u'(w_0)}\right] z \\ &= \frac{1}{2} - \frac{1}{4}\, R_R(w_0) z \end{aligned}$$

in the case of absolute and proportional bets respectively.

All in all, absolute risk aversion describes an agent's attitude towards absolute bets of plus or minus z, while relative risk aversion describes an agent's attitude towards relative bets of plus or minus zw_0, where z now is a fraction of total wealth.

Examples of commonly used utility functions

Suppose that the agent utility function takes the form

$$u(x) = -\frac{1}{\nu} e^{-\nu x} : \nu > 0$$

It follows that

$$\begin{aligned} u'(x) &= -\frac{1}{\nu} e^{-\nu x}(-\nu) = e^{-\nu x} \\ u''(x) &= -\nu e^{-\nu x} \\ &\Rightarrow \quad R_A(x) = \frac{\nu e^{-\nu x}}{e^{-\nu x}} = \nu \end{aligned}$$

which implies that the utility function displays Constant Absolute Risk Aversion (CARA). Under CARA a change in initial wealth does not affect the willingness to insure (as measured by the risk premium). In other words, changes to income/wealth will not affect an agent's economic decisions.

Suppose now that the utility function takes the form[8]:

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma} : \gamma > 0$$

In turn:

$$\begin{aligned} u'(x) &= x^{-\gamma} \\ u''(x) &= -\gamma x^{-\gamma-1} \\ &\Rightarrow R_R(x) = \frac{xu''(x)}{u'(x)} = \frac{x\gamma x^{-\gamma-1}}{x^{-\gamma}} = \gamma \end{aligned}$$

so this utility function displays Constant Relative Risk Aversion (CRRA).

Alternative CRRA specifications would be:

$$u(c_t, 1-h_t) = \frac{\left[c_t^{1-\alpha}(1-h_t)^{\alpha}\right]^{1-\gamma}}{1-\gamma}$$

where $\frac{1}{\gamma}$ is the intertemporal elasticity of substitution. Also:

$$u(c_t, 1-h_t) = \ln\left(c_t - \alpha\frac{h_t^{1+\gamma}}{1+\gamma}\right)$$

or

$$u(c_t, 1-h_t) = \ln c_t + \gamma \ln(1-h_t)$$

[8]Note that when $\lim_{\gamma\to 1} u(x) = \ln x$.

Harmonic Absolute Risk Aversion (HARA)

As an aside, note that both CARA and CRRA are particular cases of the more general concept of Harmonic Absolute Risk Aversion (HARA).

Definition 14 *HARA: A function displays HARA if the inverse of its absolute risk aversion is linear in wealth.*

Alternatively, an utility function exhibits HARA if its absolute risk aversion is a hyperbolic function:

$$R_A(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{ax+b}$$

which implies that the agent's risk tolerance is linear in wealth x. The greater the wealth, the more risk an individual would be willing to bear. The HARA class of utility functions take the following special form:

$$u(x) = \epsilon \left(\eta + \frac{x}{\gamma}\right)^{1-\gamma}$$

These functions are defined on the domain of x such that $\eta + \frac{x}{\gamma} > 0$, for which we then have:

$$\begin{aligned}
u'(x) &= \epsilon \left(\frac{1-\gamma}{\gamma}\right)\left(\eta + \frac{x}{\gamma}\right)^{-\gamma} \\
u''(x) &= -\epsilon \left(\frac{1-\gamma}{\gamma}\right)\left(\eta + \frac{x}{\gamma}\right)^{-\gamma-1} \\
R_A(x) &= \left(\eta + \frac{x}{\gamma}\right)^{-1} \\
R_R(x) &= x\left(\eta + \frac{x}{\gamma}\right)^{-1}
\end{aligned}$$

Last, note that the above becomes CARA if $\gamma = 0$ which implies $R_A = \frac{1}{\eta}$ constant, and CRRA if $\eta = 0$ as $R_R = x\frac{1}{\eta}$.

Prudence and precautionary saving

Finally, we can ask the question: what happens to savings when uncertainty about future income increases? The convexity of the marginal utility ($u''' > 0$) is called prudence and is a property of preferences, like risk aversion. Risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function. A more formal definition:

Definition 15 *Prudence: If the marginal utility is convex $u''' > 0$, then the individual is "prudent" and a rise in future income uncertainty leads to a rise in current savings.*

Prudence induces saving in order to take precaution against possible negative realizations of the income shocks. Savings induced by prudence are usually referred to as "precautionary savings".

Consider the simple two-period consumption-saving problem:

$$\begin{aligned}
\max_{(c_0,c_1,a_1)} U &= u(c_0) + \beta E_0[u(c_1)] \\
s.t. &: y_0 = c_0 + a_1 \\
&: c_1 = Ra_1 + y_1
\end{aligned}$$

where $R = 1 + r$. Assume that period-0 (y_0) income is given, while income next period income (y_1) is stochastic. Let:

$$y_1 = \bar{y} + z$$

where $\bar{y}$ is the mean and z is the random component with $E(z) = 0$ and $Var(z) = \sigma_z$. To simplify the algebra, assume that $\beta R = 1$. The Euler equation would be:

$$u'(y_0 - a_1) = E[u'(Ra_1 + y_1)]$$

which represents one equation and one unknown: a_1. Note that the LHS is increasing in a_1 since $u'' < 0$ (as $\uparrow a_1 \implies \downarrow c_0 \implies \uparrow u'(.)$), and the RHS is decreasing for the same reason, hence a_1^* is uniquely determined. Note that current consumption c_0 is determined by the period-zero budget constraint ($c_0 = y_0 - a_1$) hence a rise in savings a_1 leads to a fall in consumption.

What happens to optimal savings at $t = 0$ if the uncertainty over income next period rises, i.e., as future income becomes more risky? Consider a mean-preserving spread of z: a rise in the variance of z that keeps the mean constant. The Euler equation would then be:

$$u'(y_0 - a_1) = E[u'(Ra_1 + \bar{y} + z)]$$

The equation shows that if $u'(.)$ is convex then, by Jensen's inequality, a mean preserving spread of z will increase the value of the RHS, inducing a rise in a_1^* and a movement in the LHS that restores the equilibrium. All in all savings increase.

To understand how Jensen's inequality was used, recall that the inequality stated that for a given random variable y, if $f(.)$ is a convex function then:

$$E[f(y)] \geq f(E[y])$$

and the opposite holds if the function is concave. Start from a situation where the $Var(x) = 0$ and $y_1 = \bar{y}$. The RHS of the Euler equation becomes:

$$E[u'(Ra_1 + \bar{y})] = u'(Ra_1 + \bar{y})$$

since all is deterministic. Now add some uncertainty in the sense that $Var(z) > 0$ and the RHS of the

Euler equation becomes

$$\underbrace{E[u'(Ra_1 + \bar{y} + z)]}_{\text{RHS under uncertainty}} > u'(E[Ra_1 + \bar{y} + z]) = \underbrace{u'(Ra_1 + \bar{y})}_{\text{RHS under certainty}}$$

where the first inequality follows from Jensen's inequality and the convexity of the utility function, and the second equality from the fact that $E(z) = 0$. Hence, relative to the certainty case, higher volatility will increase the agent's demand for savings, increasing the optimal level of a_1^* and depressing current consumption c_0.

12 - Model Calibration:

Kydland and Prescott suggest a way to identify parameters so that the model simulated output can match certain data moments. The basic idea is:

- Use microeconomic studies or theory to pin down the model's parameter values
- Solve the model numerically and simulate the economy
- Compare moments (standard deviations, correlations, etc) of the simulated economy with those of statistical agencies (i.e.: the "real" economy)

A few examples:

The discount factor: β. Remember that $\beta = \frac{1}{1+\rho}$ where ρ is the agent's subjective discount factor. In a standard model with capital markets we would have that at the non-stochastic steady state $\beta = \frac{1}{1+\rho} = \frac{1}{R_t}$ where R_t is the gross interest rate. Further, the average interest rate in the US is usually around 4%, which is about 1% quarterly. We therefore have that $0.01 = R - 1 = \frac{1}{\beta} - 1 \rightarrow \beta = 0.99$. Cooley and Prescott[9] follow a slightly more complex process and calibrate β as 0.987.

[9] Thomas Cooley's *"Frontiers of Business Cycle Research"*, Chapter 1.

The depreciation rate: δ. Cooley and Prescott estimate that for the US the depreciation rate is 4.8% annually, so about 1.2% per quarter ($\delta = 0.012$).

The share of income: α. A common production function used is the Cobb-Douglass specification: $y_t = k_t^{\alpha} n_t^{1-\alpha}$. Remember that α will be capital's share of output and $1-\alpha$ that of labor. For the US this quantity can be estimated from NIPA accounts. Cooley and Prescott set α to 0.40.

The relative risk aversion coefficient: γ. A common type of CRRA utility function:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

where $1/\gamma$ is the intertemporal elasticity of substitution. Also, when $\gamma = 0$ utility is linear, when $\gamma = \infty$, utility is Leontief, and when $\gamma = 1$ utility becomes logarithmic. Micro studies have estimated γ to be close to 1 so many researchers choose to use this value to calibrate the parameter.

The persistence on the TFP proess: ρ. If the model is stochastic chances are it will include the laws of motion for some exogenous disturbances. Perhaps the most common aggregate disturbance you might currently encounter is a technological shock. In models with perfect competition and constant returns to scale, the parameters of the TFP process are easy to estimate. Once we've built the TFP data series (for e.g.: as as $z_t = \frac{y_t}{k_t^{\alpha} n_t^{1-\alpha}}$)[10] we can assume this process follows an autoregressive structure (say an AR(1)) and estimate the parameters of interest (mainly ρ and σ_ϵ).

[10]Data on US TFP can also be found on FRED from the STL Fed.