Notes on Growth Theory

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An economy's ability to deliver improving standards of living crucially depends on its long run rate of economic growth. Small differences in growth rates compounded over many years can have lasting effects on individuals' incomes. Growth theory is an attempt at identifying the forces that determine the pace at which an economy grows over long periods of time. Along with changes in productivity, saving and investment decisions play a central role in the analysis. In turn the rate at which capital goods are accumulated will be an important factor determining the standard of living that a country's citizens can attain.

This note describes some of the most influential works on neoclassical growth theory. We begin exploring the seminal works of Robert Solow and Frank Ramsey. The first model effectively exemplifies the process of capital accumulation under a constant saving rate, while the second one achieves the same in a context where the saving rate is optimally chosen every period to maximize an agent's utility. Because in both models the main source of long-run growth resides in a persistent technological progress (not explained by either one), these type of models are often categorized under exogenous growth theory.

Endogenous Growth theory, on the other hand, seeks to solve the above mentioned limitation by constructing models in which the long-run growth rate of an economy is determined by incentives or other "fundamental" considerations. For example, firms in some of this type of models will engage in R&D activities because of the economic rents they might be able to extract once granted a patent. Other models seek to capture the effects of human capital accumulation in long-run productivity gains as well as the effects of production externalities. Models in this area include the AK model as well as Mankiew-Romer's human capital model .

*DISCLAIMER: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even somewhat incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!

Exogenous Growth Theory

1 The Solow Model

The Solow model is a useful tool to study the dynamics of economic growth. The model captures an economy as it evolves over time and describes the process of capital accumulation.

The general assumptions are:

1. Time is discrete.
2. Perfect competitive markets
3. No externalities
4. Complete information environment
5. Closed Economy and no Government. In turn: $Y_t = C_t + I_t$.
6. Single good economy, used for both consumption and investment.
7. Neoclassical production technology: $Y_t = A_t F(K_t, N_t)$ where:
   (a) $F(K_t, N_t)$ experiences constant returns to scale (i.e.: homogeneous of degree 1)
   (b) $F(K_t, N_t)$ experiences diminishing marginal returns on both inputs
   (c) $F(K_t, N_t)$ is twice differentiable and strictly increasing
8. Exogenous and constant savings rate $''s'' : 0 < s < 1$.
9. Exogenous and constant depreciation rate $''\delta'' : 0 < \delta < 1$.
10. Workforce is a constant share of the population[1]
11. Physical capital accumulates following: $K_{t+1} = (1-\delta)K_t + I_t$

Given the assumptions above note that:

1. Since $Y_t = C_t + I_t$, in equilibrium, $S_t = I_t$.
2. Investment will be used for: (a) restoring depreciated capital, (b) increasing the existing capital stock. In turn: $I_t = K_{t+1} - (1-\delta)K_t$
3. $F_K(K_t, N_t)$ and $F_N(K_t, N_t)$ will be increasing and strictly concave

[1]This implies that if the population grows, so will the labor force.

1.1 Solution

To solve this model Solow did not set up a general equilibrium problem with optimizing individuals and firms, but rather, he directly specified an agent's decision rules. As such, we have that:

1. Individuals does not value leisure and inelastically supply all their labor endowment
2. Agents will consume (and save) a fix proportion of income every period: $c_t = (1-s)y_t$, $s_t = sy_t$
3. National savings are always a constant proportion of aggregate income: $S_t = sY_t$

1.2 Versions

There are at least three important version of this model:

1. Model with no population and no persistent technological growth.
2. Model with population growth but no persistent technological growth.
3. Model with population and persistent technological growth.

Note that because the model seeks to understand the evolution of standards of living, all versions of the model will be written in some sort per-capita or per efficiency units.The assumption of homogeneity of degree one of the production function will be crucial to allow for this.

1.2.1 Solow model with no technology nor population growth

Let $A_t = 1\ \forall t$ and $g_n = 0\ \forall t$. The aggregate production technology then becomes:

$$Y_t = F(K_t, N_t)$$

which because of homogeneity of degree 1, the above can be written as:

$$\frac{Y_t}{N_t} = F\left(\frac{K_t}{N_t}, 1\right)$$

Sine we are interested in improvements in standards of living, output and consumption per-capita are better proxies for the degree of living conditions provided by an economy. Let $y_t = \frac{Y_t}{N_t}$ and $k_t = \frac{K_t}{N_t}$ be per-capita output and per-capita capital stock. Rewrite the above as:

$$y_t = f(k_t)$$

We begin with $k_0 > 0$, with which the economy produces y_0. A fraction s of output is saved $s_0 = sy_0$, and the rest consumed $c_0 = (1-s)y_0$. Because for a closed economy in equilibrium $S = I$, s_0 will be used to make up for depreciation of the existing capital δk_0 and to accumulate more capital which leads to $k_1 > k_0$. In turn k_1 generates y_1 and the process repeats itself, each time obtaining a greater capital stock (... $k_2 > k_1 > k_0$), output (... $y_2 > y_1 > y_0$) and consumption (... $c_2 > c_1 > c_0$) per capita. This is the process of capital accumulation which drives output growth and with it, improvements in standards of living. Schematically:

$$
\begin{tikzcd}
k_0 \arrow[r] & y_0 \arrow[r] \arrow[dr] & c_0 & & & & \\
& & s_0 = I_0 \arrow[r] \arrow[dr] & \delta k_0 & & & \\
& & & k_1 \arrow[r] & y_1 \arrow[r] \arrow[dr] & c_1 & \\
& & & & & s_1 = I_1 \arrow[r] & \cdots
\end{tikzcd}
$$

When does this process of capital accumulation end? Intuitively the answer is when there are no more resources left to further increase the capital stock. In other words, the accumulation ends when the available savings are just enough to replenish the physical capital that is worn out during the production process that period. The economy then reaches a point known as the steady state where output, consumption, savings, etc. no longer grow. The economy will remain at the steady state indefinitely unless something reignites the capital accumulation process.

It should be noted that in this particular model specification both aggregate and per-capita variables will seize to grow once they reach their steady state value. Also, note that different parameter values for δ or s, will yield different steady state values.

INSERT GRAPH

Last, note that a one-time technological improvement (e.g.: $A_t = 1$ changes to $A_t = 2$) will push the steady state further to the right (i.e.: towards a higher per-capita capital stock and output) and reignite the capital accumulation process. Once the new steady state is achieved, however, the capital accumulation process stops and economic growth comes to an end.

This, unfortunately, is not entirely in line with what we usually observe in reality. Economies do not just stop growing and remain at the same income level forever. That is why a sequence of technological innovations is required to explain continuous economic growth if the world is viewed through the eyes of the model. In other words, sustained growth requires technological progress in the Solow framework. Capital accumulation alone cannot, by itself, sustain long-term growth.

1.2.2 Solow model with no technology and population growth

Let $A_t = 1\ \forall t$ but the growth rate of population $g_n > 0\ \forall t$. In turn, the population at time t will be: $N_t = (1 + g_n)^t N_0$ where N_0 is some initial population. Given that the population now grows systematically, the model is no longer stationary [2]. Luckily rewriting the model in per-capita terms like we did before solves the stationarity issue, yet brings a new interesting dimension of analysis into play.

Since population is grows deterministically, this implies that at the steady state resources will need to be allocated not only to make up worn out capital, but also to equipping the new workers entering the labor force each period. That is why per-capita variables will be constant, while aggregate ones won't. For example, per-capita output will be constant at steady state while aggregate output will continue to rise fueled by the expansion in population and the number of new workers associated with it. In other words, aggregate output will continue to rise at the rate g_n. This is in contrast to the previous model where both per-capita and aggregate variables ceased to grow at the steady state.

[2] Technically, the population growth induces a deterministic trend which rends the model non-stationary

1.2.3 Solow model with technology and population growth

Let the growth rate of technology $g_A > 0 \ \forall t$ and the growth rate of population $g_n > 0 \ \forall t$. In turn, the technology level at time t will be $A_t = (1+g_A)^t A_0$ where A_0 is the initial technology level, while the population at any given point is: $N_t = (1+g_n)^t N_0$. Both technology and population have a deterministic trend, which again forces us to rewrite the model under a stationary inducing transformation. The model is rewritten in effective units of labor, since technological progress may reduce the number of workers needed to achieve a given level of output. Just like before:

$$Y_t = F(K_t, A_t N_t)$$

which because of degree 1 of homogeneity can be written as:

$$\frac{Y_t}{A_t N_t} = F\left(\frac{K_t}{A_t N_t}, 1\right)$$

Let $y_t = \frac{Y_t}{A_t N_t}$ and $k_t = \frac{K_t}{A_t N_t}$ be output and capital stock per effective worker or per effective unit of labor. Rewrite the above as:

$$y_t = f(k_t)$$

Like in the previous version per-effective worker variables will cease to grow at the steady state while aggregate ones will continue to do so, even if at different rates. For example, aggregate labor will grow at the rate g_n, while aggregate output at the rate $g_A + g_n$. A useful summary below:

	Steady State rate of growth
Capital per effective worker: k_t	0
Output per effective worker: y_t	0
Capital per worker	g_A
Output per worker	g_A
Aggregate Labor	g_n
Aggregate Capital	$g_A + g_n$
Aggregate Output	$g_A + g_n$

2 The Ramsey Model

Ramsey (1928) followed much later by Cass and Koopmans (1965) formulated one of the basic workhorse models in modern macroeconomics. The Ramsey-Cass-Koopmans model (colloquially referred to as the Ramsey model or the Neoclassical growth model[3]) explicitly models consumer behavior and endogenizes its savings decision.

The model originally stipulated a social planner who would directly allocate resources to maximize an agent's lifetime utility. One could also assume a decentralized set up where households earn wages and capital rental income, yet in an attempt to stay as close as possible to the original formulation (and also to the Solow model) we'll present the social planner set up. Perhaps the one concession will be that while the model was originally conceived in continuous time, this note time will be assumed to be discrete.

The main assumptions are:

1. Closed economy: $Y_t = C_t + I_t$
2. No Government
3. Time is discrete
4. One representative consumer with infinite lifetime: $U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$
5. Population growth rate is exogenous and constant: $N_t = (1 + g_n)N_{t-1}$
6. Labor supply will (following population) increase exogenously at a constant rate
7. The economy has a perfectly competitive production sector that uses Cobb-Douglass production technology : $Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$
8. Capital accumulation is as in the Solow model: $k_{t+1} = (1-\delta)k_t + i_t$
9. Exogenous and constant depreciation rate δ

where $\beta \in (0,1)$ and the flow utility or period utility function is assumed to be strictly increasing ($u'(c) > 0$) and strictly concave ($u''(c) < 0$). In addition we will impose an Inada conditions: $\lim_{c \to 0} = \infty$ and $\lim_{c \to \infty} = 0$

Note that:

1. The technological constraints are identical to those in the Solow model. Output is produced using labor and capital with standard neoclassical production function.
2. Population and technology are assumed to be constant. However, technology and labor force growth can be accommodated in this model by applying the normalization of rewriting everything in terms of "effective labor units" as in the Solow model:

$$\begin{aligned} \frac{Y_t}{A_t N_t} &= \frac{K_t^{\alpha} A_t^{1-\alpha} N_t^{1-\alpha}}{A_t N_t} \\ &= \left(\frac{K_t}{A_t N_t} \right)^{\alpha} \\ &\Rightarrow \quad y_t = k_t^{\alpha} \end{aligned}$$

[3] Sometimes also referred to as neoclassical growth model with endogenous saving rate.

3. Note that A_t can accommodate many types of specifications:

 - If $\{A_t\}$ is known or constant, then the growth model is said to be deterministic.
 - If $\{A_t\}$ follows some random process, then the growth model is said to be stochastic

Given these constraints, the social planner selects a consumption/savings plan to maximize the consumer's utility. Formally, the problem is to pick a sequence (stochastic process) $\{\hat{c}_t\}$ to maximize[4]:

$$\begin{aligned}
\max_{\{c_t,i_t\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. &: f(k_t) \geq c_t + i_t \\
&: i_t = k_{t+1} - (1-\delta)k_t \\
&: c_t, k_t \geq 0 \\
&: k_0 > 0 \text{ given}
\end{aligned}$$

which can be re-written as:

$$\begin{aligned}
\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. &: f(k_t) \geq c_t + k_{t+1} - (1-\delta)k_t \\
&: c_t, k_t \geq 0 \\
&: k_0 > 0 \text{ given}
\end{aligned}$$

the lagrangean for this problem is:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [f(k_t) - c_t - k_{t+1} + (1-\delta)k_t] \right\}$$

and the corresponding first order conditions:

$$\begin{aligned}
\frac{\delta L}{\delta c_t} &= 0 \iff u'(c_t) = \lambda_t \\
\frac{\delta L}{\delta k_{t+1}} &= 0 \iff \beta \lambda_{t+1} [f'(k_{t+1}) + (1-\delta)] = \lambda_t
\end{aligned}$$

substitution yields the Euler equation:

$$u'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = u'(c_t)$$

The Euler equation has an intuitive economic interpretation: along an optimal path the marginal utility from consumption at any point in time is equal to its opportunity cost. The Euler equation is a necessary condition for an optimal policy. Another necessary condition is the transversality condition (TVC), which implies that the present value of future capital k_t must tend to zero:

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0$$

[4]Note that in this specification we have assumed $\{A_t\}$ to be known or constant

Together, the Euler equation and the capital accumulation equation represent a second order difference equation:

$$
\begin{aligned}
u'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] &= u'(c_t) \\
f(k_t) - c_t &= k_{t+1}
\end{aligned}
$$

In order to have a unique solution, a second-order difference equation must be supplemented with exactly two boundary conditions. The first boundary condition is the initial value of capital k_0. If we would somehow know the starting value of consumption c_0 our problem would be done. Unfortunately we do not and we'll have to find our second boundary condition elsewhere. In this particular case, the boundary condition will be given by the transversality condition. It will turn out that there is only one possible value of c_0 that doesn't violate the TVC.

Endogenous Growth Theory

A key issue with the type of models just described is that in the absence of technological growth, output per capita will eventually converge to a fixed level (i.e.: its steady state level). Without exogenous technological progress, diminishing returns always catch up in the long run and output growth stalls. This is why critics claim that this type of models are not really growth models, but rather models *about* growth as they assume rather than explain the behavior of the crucial determinant of long-run GDP growth rate.

Endogenous growth models have tried to generate long-run growth without resorting to exogenous technological change. These models try to explain improvements in productivity (and hence in the long-run growth of output) *endogenously* or within the model. **The basic idea in most of these models is to do away with decreasing returns to scale in the production technology**.

A number of reasons have been provided to explain why, for the economy as a whole, the marginal product of capital may not be diminishing. One explanation emphasizes the role of human capital. As economies accumulate capital and become wealthier, it becomes feasible to devote more resources to things like improved nutrition, schooling and health care. This investment in people increase the country's human capital, which in turn raises productivity. If the physical capital stock increases while the stock of human capital remains fixed, we would again experience diminishing marginal productivity of physical capital as un Solow.

A second rationalization of a constant marginal productivity of capital is based on the observation that, in a growing economy, firms have incentives to undertake research and development (R&D) activities. These activities increase the stock of commercially valuable knowledge, including new products and production technologies. Increases in capital and output tend to generate increases in technical knowledge and the resulting productivity gains offset any tendency for the marginal productivity of capital to decline.

All in all, the advantages of endogenous growth theory seem twofold. First, the literature attempts to explain (rather than assume) an economy's long-run productivity growth. Second, it shows how the long-run growth rate of output may depend on factors such as the country's saving rate which can be in turn affected by government policies. Below I describe two canonical models in this literature.

3 The AK Model

The AK model is the simplest possible endogenous growth model. In this case, the production possibilities is assumed to be:

$$Y_t = AK_t$$

where $A > 0$ is a positive constant. Output per capita is now:

$$\begin{aligned} f(k_t) &= Ak_t \\ &\Rightarrow A_t = \frac{f(k_t)}{k_t} \end{aligned}$$

Several of the model's assumption are identical to those of made by Solow. The economy is closed, population will be constant and there is no government. Additionally, as in the Solow model, savings will be an exogenous and constant share of output. From the familiar law of motion of capital:

$$\begin{aligned} i_t &= k_{t+1} - (1-\delta)k_t \\ &\Rightarrow k_{t+1} - k_t = i_t - \delta k_t \\ &\Rightarrow k_{t+1} - k_t = sf(k_t) - \delta k_t \\ &\Rightarrow \frac{k_{t+1} - k_t}{k_t} = s\left(\frac{f(k_t)}{k_t}\right) - \delta \end{aligned}$$

Replacing the into the above yields an expression for the growth rate of capital:

$$\frac{k_{t+1} - k_t}{k_t} = sA - \delta$$

Finally because output is proportional to the capital stock, the growth rate of output equals the growth rate of the capital stock:

$$\frac{\Delta y}{y} = sA - \delta$$

Which implies that as long as $sA > \delta$, the economy will have positive long-run growth without any technological progress. The equation also suggests that the rate of growth depends on the saving rate s. As the number of workers is assumed to be constant, the growth rate of output per worker equals the growth rate of output and thus depends on the saving rate.

The intuition behind savings affecting long-run growth in this framework is that higher rates of saving and capital formulation stimulate greater investment in human capital and R&D. The resulting increases in productivity help to spur long-run growth. This result stands in sharp contrast to the results of the Solow model, in which the saving rate does not affect the long-run growth rate of an economy.

4 The Mankiew-Romer-Weil Model of human capital

This model extends the basic Solow model by explicitly incorporating the role of human capital into the production function. Human capital is accumulated from saving output, just as with physical capital. Think of this as bypassing current consumption in order to increase future returns on labor, basically going to school. In fact, schooling rates will be used as proxies for saving rates in human capital in the original version of the model.

Assumptions:

1. Output is produced by a homogenous of degree one production function: $Y_t = F(K_t, H_t, A_t N_t)$
2. Inputs for production are: physical capital K_t, human capital H_t and effective labor $A_t N_t$.
3. Exogenous and constant saving rates for human and physical capital: s_h, s_k
4. Exogenous and constant depreciation rates for human and physical capital: δ_h, δ_k

5. The two types of capital accumulate following:

$$K_{t+1} = (1 - \delta_k) + I_{k,t}$$
$$H_{t+1} = (1 - \delta_h) + I_{h,t}$$

As previously done, we can put the system in stationary form by normalizing all variables by $A_t N_t$. The stationary production function now takes the form:

$$y_t = f(k_t, h_t)$$

Utilizing the normalized form of the capital accumulation equations, and the constant saving rate in each type of capital we can proceed to obtain a dynamic system in stationary form, just like we did in the Solow model:

$$k_{t+1} - k_t = s_k f(k_t) - \delta_k k_t$$
$$h_{t+1} - h_t = s_h f(h_t) - \delta_h h_t$$

As in the one sector case, the steady states are easy to find from here.