Notes on Overlapping Generations Models

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This note describes another major workhorse model of modern macroeconomics: the Overlapping Generations (OLG) model. The classic references are Allais (1947), Samuelson (1958) and Diamond (1965). Unlike the Ramsey model the OLG model assumes that there is turnover in the population. That is to say, rather than a fixed number of infinitely lived agents, new individuals are continually being born, and old individuals dying.

This variation will introduce several differences with a model of infinitely lived agents. Mainly:

- A competitive equilibrium maybe Pareto suboptimal
- There may exist a continuum of equilibria
- Outside money might have positive value

1 The General Model

Let us describe the model formally now. Its fundamental assumptions are:

1. Time is discrete
2. There is a single consumption good
3. Agents live only for two periods
4. Every period N_t individuals are born
5. Population is assumed to grow at rate $g_n : N_{t+1} = (1 + g_n)N_t$.
6. At all times there is a certain share of "young" (N_t) and "old" $\left(\frac{N_t}{1+g_n}\right)$agents.
7. When young, each individual inelastically supplies 1 unit of labor and receives wage w_t.
8. When old, each individual exits the labor market and lives off investment income.
9. No need for a transversality condition
10. No need for a borrowing limit (i.e.: No Ponzi game condition)

* **DISCLAIMER**: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even somewhat incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!

Remark 1 *Notation can cause a lot of confusion in the OLG model. That is because at every point in time, there are two generation of agents taking decisions.*

The are two commonly used notation approaches. The first one is to classify agents according to what generation they belong to. As such c_t^t would represent the consumption at time t of an individual born at time t, while c_{t+1}^t would represent that same agent's consumption in the following period. The first case corresponds to the period when that generation is considered to be young, while the second when it is considered to be old. The main advantage of this notation is that it allows us to keep track of exactly what generation is doing what at every point in time. The table below exemplifies this notation style:

Generation

Period	$t-1$	t	$t+1$	$t+2$
t	c_t^{t-1}	c_t^t		
$t+1$		c_{t+1}^t	c_{t+1}^{t+1}	
$t+2$			c_{t+2}^{t+1}	c_{t+2}^{t+2}
$t+3$				c_{t+3}^{t+2}

The second approach utilizes the recursive nature of the economy and focuses on whether an individual is considered to be young or old at that particular point in time. Let "1" represent the young generation, while "2" the old one. In that case c_t^1 would represent the time t consumption of an agent born in that period, while c_t^2 the time t consumption of an individual born in the previous one. This notation style focuses on what is more relevant to the model: mainly what is each generation type doing every time period.

Generation

Period	I	II	III	IV
t	c_t^2	c_t^1		
$t+1$		c_{t+1}^2	c_{t+1}^1	
$t+2$			c_{t+2}^2	c_{t+2}^1
$t+3$				c_{t+3}^2

Both methods are closely related as they essentially are seeking to keep track of the same object. I will follow the second approach when writing this note.

1.1 Endowment Economy

We begin with the most basic set-up, where we abstract from any production decision. Each time period every individual will receive an endowment of consumption goods. Let (e_t^1, e_t^2) denote the young and old generations' endowments received at time t. Additionally, let (e_t^1, e_{t+1}^2) denote a generation's endowment of the consumption good in the first and second period of their life and (c_t^1, c_{t+1}^2) denote their corresponding consumption levels. Last in period 1, there is an initial old generation 0 that receives endowment e_0^2 and consumes c_0^2.

1.1.1 Households

Lifetime household preferences are assumed to be represented by an additively separable utility function of the form:

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2)$$

and the preferences of the initial old generation is represented by

$$U_0 = u(c_0^2)$$

We shall assume that u(.) is strictly increasing, strictly concave and continuously differentiable. Let r_{t+1} be the interest rate from period t to $t+1$ and let s_t be the savings of the young generation in period t in units of the consumption good. It is assumed that the older generations have no incentives to save past beyond their life (i.e.: there is no bequest motive). Assuming a sequential market structure, the equilibrium can be defined as:

Definition 1 *Given s_{-1}, an equilibrium is an allocation $(\hat{c}_0^2, \{\hat{c}_t^1, \hat{c}_t^2, \hat{s}_t\}_{t=1}^{\infty})$ and interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ such that:*

1. *Given $\{r_t\}_{t=1}^{\infty}$ for each $t \geq 1$, $(\hat{c}_t^1, \hat{c}_t^2, \hat{s}_t)$ solves:*

$$\begin{aligned}\max_{\{s_t, c_t^1, c_{t+1}^2\}} U_t &= u(c_t^1) + \beta u(c_{t+1}^2) \\ s.t. &: e_t^1 \geq c_t^1 + s_t \\ &: e_{t+1}^2 + (1 + r_{t+1}) s_t \geq c_{t+1}^2\end{aligned}$$

2. *Given r_1, c_0^2 solves:*

$$\begin{aligned}\max_{c_0^2} U_0 &= u(c_0^2) \\ s.t. &: e_0^2 + (1 + r_1) s_{-1} \geq c_0^2\end{aligned}$$

3. *Goods market clears:*

$$e_t^1 + e_t^2 = c_t^1 + c_t^2$$

Note that when we wrote down the sequential formulation of equilibrium for an infinitely lived representative agent model we needed to add a short-sale constraint on borrowing in order to prevent Ponzi schemes: the continuous rolling over of higher and higher debt. This is not necessary in the OLG model as people live for a finite (two) number of periods.

1.2 Production Economy

We now describe a set-up where household accumulate capital and rent out labor services, and firms engage in production decisions.

1.2.1 Households

Let c_t^1 and c_t^2 denote the consumption levels of young and old agents at time t. In turn, an agent's lifetime utility will depend on c_t^1 and c_{t+1}^2:

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2)$$

Each individual is born with no capital or bond holdings. When young, each agent inelastically supplies one unit of labor to the market and receives w_t. It can choose to consume or save (buy capital):

$$w_t \geq c_t^1 + k_{t+1}$$

When old, agents retire and live off their investment income:

$$r_{t+1}k_{t+1} \geq c_{t+1}^2$$

To complete the set-up, at time zero there is an old generation (worker -1) with an exogenous capital stock $k_0 > 0$. Note that:

1. The only variable with a superscript to denote generation is consumption, since only the young work and invest, and only the old receive investment income.
2. The old generation are not allowed to accumulate capital (i.e.: save).
3. The depreciation rate is 100 %. This is for two reasons: first, if the depreciation rate was lower, we would have to specify how capital is passed on from the old to the young; and second, that since each period is supposedly an entire generation of 30-40 years, 100 % seems adequate.

1.2.2 Firms

There are many firms, each with production function $y_t = F(k_t, n_t)$ satisfying CRS. Markets are competitive, thus labor and capital will earn their marginal products and firms make zero profits.

$$\begin{aligned} \max_{k_t, n_t} \pi &= y_t - r_t k_t - w_t n_t \\ s.t. &: y_t = F(k_t, n_t) \end{aligned}$$

where $F(k_t, n_t)$ is a neoclassical production function.

1.2.3 Equilibrium

There is some initial capital stock k_0 that is owned equally by all old individuals. Therefore at $t = 0$ the capital owned by the old and the labor supplied by the young are combined to produce output. The old consume both their capital income and their existing wealth. The young divide their labor income w_t between consumption and saving. They carry their savings forward to the next period in the form of capital. This capital is then combined with the labor supplied by the next generation young and so on.

An equilibrium is a sequence of prices $\{r_t, w_t\}$ and allocations $\{\hat{c}_t^1, \hat{c}_t^2 \hat{k}_t, \hat{n}_t\}$ such that:

1. Taking prices as given, the allocations $(\hat{c}_t^1, \hat{c}_{t+1}^2, \hat{k}_{t+1})$ solve worker t's utility maximization problem.
2. Taking prices as given, the allocations $\hat{k}_t$ and $\hat{n}_t$ solve the firm's profit maximization problem.
3. All markets clear:

$$\begin{aligned} n_t &= 1 \\ c_t^1 + c_t^2 + k_{t+1} &= F(k_t, 1) \end{aligned}$$

1.2.4 Solving the Model

The Lagrangean for the worker t's utility maximization problem is:

$$L = u(c_t^1) + \beta u(c_{t+1}^2) + \lambda_t[w_t - c_t^1 - k_{t+1}] + \theta_t[r_{t+1}k_{t+1} - c_{t+1}^2]$$

The first order conditions are:

$$\begin{aligned} u'(c_t^1) &= \lambda_t \\ \beta u'(c_{t+1}^2) &= \theta_t \\ \theta_t r_{t+1} &= \lambda_t \end{aligned}$$

Putting the first order conditions together one may obtain:

$$\beta u'(c_{t+1}^2) r_{t+1} = u'(c_t^1)$$

which looks very similar to an Euler equation derived from an infinitely lived representative agent model. Notice, however, that the consumption levels are for a particular generation, rather than a representative agent. Or perhaps one could think the representative agent of a particular generation.

The firms firms order conditions are:

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= y_t - r_t k_t \end{aligned}$$

2 Examples and Applications

The OLG framework can be used to study many interesting topics ranging from bequests to social security issues. Below are some examples.

2.1 Social Security

N_t individuals are born in period t and live for two periods, working and saving in the first and living off capital in the second period. Assume population is growing at a constant rate n and there is no technological progress $g = 0$ (so that we can normalize $A = 1$). Markets are competitive and labor and capital are paid their marginal products. There is no capital depreciation ($\delta = 0$). Utility is logarithmic and we assume that individual's discount rate is zero (i.e.: $\rho = 0$). Lifetime utility can be written as:

$$U = \ln(c_t^1) + \beta \ln(c_{t+1}^2)$$

The production function in per capita terms is:

$$y_t = f(k_t) = k_t^\alpha$$

The government taxes each young individual a lump sum T. It allocates a fraction γ of this tax to an Individual Retirement Account which will provide the individual with $(1 + r_{t+1})\gamma T$ when old. The remaining fraction $(1 - \gamma)$ is used to pay retirement benefits to the current old generation. Therefore, when an agent retires, the individual also receives a State benefit of $(1 + n)(1 - \gamma)T$.

2.1.1 Questions:

1. What kind of social security system is this? Is it a pay as you go (i.e.: unfunded) or a capitalized (i.e.: fully funded) system? Also explain why $(1 + n)$ appears in the State's benefit retirement expression.
2. Write down the budget constraint faced by an individual at time t.
3. Use the budget constraint and the individual's lifetime utility function to solve for first period consumption c_t^1 and first period savings (s_t).
4. Use the savings function previously derived and the production function to determine the relationship between k_{t+1} and k_t.

2.1.2 Answers:

(1)
The system is partially unfunded (pay as you go) and partially funded (capitalized). There is a fraction γ of taxes that is invested and used to pay retirements benefits to the current young generation when they reach retirement age at $t + 1$. The remaining fraction is used to pay benefits to the current old generation. A fully pay as you go system would have $\gamma = 0$ and a fully funded system would have $\gamma = 1$. The rate of population growth $(1 + n)$ appears in the benefit formula because the benefits paid out by the State will grow at this same rate.

(2)
The constraints are:

$$\begin{aligned} w_t &\geq c_t^1 + s_t + T \\ (1 + r_{t+1})s_t + (1 + r_{t+1})\gamma T + (1 + n)(1 - \gamma)T &\geq c_{t+1}^2 \end{aligned}$$

At equilibrium, substituting s_t yields:

$$(1+r_{t+1})(w_t - c_t^1) + (1+r_{t+1})\gamma T + (1+n)(1-\gamma)T \quad = \quad c_{t+1}^2$$

and after some rearranging:

$$(1+r_{t+1})(w_t - c_t^1) + (1-\gamma)(n - r_{t+1})T \quad = \quad c_{t+1}^2$$

(3)

The problem's lagrangean is:

$$L = \ln c_t^1 + \ln c_{t+1}^2 + \lambda[(1+r_{t+1})(w_t - c_t^1) + (1-\gamma)(n-r_{t+1})T - c_{t+1}^2]$$

The first order conditions are:

$$\begin{aligned}
\frac{\delta L}{\delta c_t^1} &= 0 \iff \frac{1}{\hat{c}_t^1} = (1+r_{t+1})\lambda \\
\frac{\delta L}{\delta c_{t+1}^2} &= 0 \iff \frac{1}{\hat{c}_{t+1}^2} = \lambda \\
\frac{\delta L}{\delta \lambda} &= 0 \iff (1+r_{t+1})(w_t - \hat{c}_t^1) + (1-\gamma)(n-r_{t+1})T = \hat{c}_{t+1}^2
\end{aligned}$$

The Euler equation is:

$$\begin{aligned}
\frac{\hat{c}_{t+1}^2}{\hat{c}_t^1} &= (1+r_{t+1}) \\
&\Rightarrow \hat{c}_{t+1}^2 = (1+r_{t+1})\hat{c}_t^1
\end{aligned}$$

Replacing the above into the budget constraint yields:

$$\begin{aligned}
(1+r_{t+1})\hat{c}_t^1 &= (1+r_{t+1})(w_t - \hat{c}_t^1) + (1-\gamma)(n-r_{t+1})T \\
&\Rightarrow (1+r_{t+1})\hat{c}_t^1 = w_t(1+r_{t+1} - \hat{c}_t^1(1+r_{t+1}) + (1+\gamma)(n-r_{t+1})T \\
&\Rightarrow 2(1+r_{t+1})\hat{c}_t^1 = w_t(1+r_{t+1} + (1+\gamma)(n-r_{t+1})T \\
&\Rightarrow \hat{c}_t^1 = \frac{1}{2}w_t + \frac{(1+\gamma)(n-r_{t+1})T}{2(1+r_{t+1})}
\end{aligned}$$

Given that $\hat{s}_t = w_t - \hat{c}_t^1 - T$ we can find the individual's savings by substituting for optimal c_t^1:

$$\hat{s}_t = \frac{1}{2}w_t - \left[1 + \frac{(1-\gamma)(n-r_{t+1})}{2(1+r_{t+1})}\right] T$$

(4)

The total capital stock in this economy will be equal to the savings made by the consumers in the first period, plus the resources allocated to the IRA by the Government. In turn:

$$K_{t+1} = s_t N_t + \gamma N_t T$$

Since we are asked about per capita capital, we must re-write the above as:

$$\begin{aligned}
\frac{K_{t+1}}{(1+n)N_t} &= \frac{s_t N_t}{(1+n)N_t} + \frac{\gamma N_t T}{(1+n)N_t} \\
&\Rightarrow \quad k_{t+1} = \frac{s_t}{(1+n)} + \frac{\gamma T}{(1+n)} = \frac{1}{(1+n)}[s_t + \gamma T] \\
&\Rightarrow \quad k_{t+1} = \frac{1}{(1+n)}\left[\frac{1}{2}w_t - \left(1 + \frac{(1-\gamma)(n-r_{t+1})}{2(1+r_{t+1})}\right) T + \gamma T\right]
\end{aligned}$$

From the first order condition of the firm, we obtain that:

$$\begin{aligned}
w_t &= (1-\alpha)K_t^{\alpha} N_t^{-\alpha} \\
&= (1-\alpha)k_t^{\alpha}
\end{aligned}$$

Substituting the equilibrium condition above and after some algebra yields:

$$k_{t+1} = \frac{1}{(1+n)}\left[\frac{1}{2}(1-\alpha)k_t^{\alpha} - (1-\gamma)\left(\frac{r_{t+1}+n+2}{2(1+r_{t+1})}\right)\right]$$

2.2 Model's Dynamics