# Notes on Real Business Cycles *

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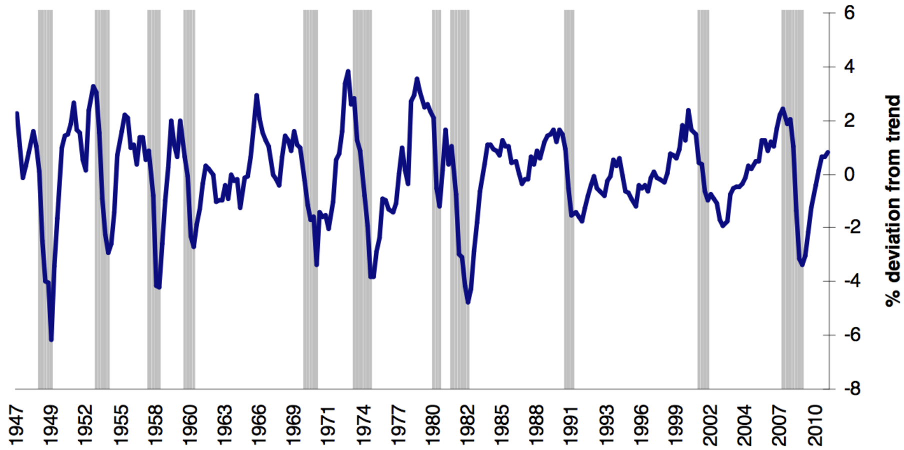
Since the Industrial Revolution, economic growth has transformed nations and greatly improved living standards. Yet even in prosperous countries, economic expansion has been periodically interrupted by episodes of declining production and income and rising unemployment. Whether brief or more extended, declines in economic activity have been followed almost invariably by a resumption of economic growth. This repeated sequence of economic expansion giving way to temporary decline followed by recovery, is known as a business cycle.

Typically, a business cycle has four very distinctive parts: a peak, a trough, an expansion and a recession. An empirical analysis of U.S. time series shows that movements in all the aggregate variables of interest (output, consumption, investments, etc.) often happen around a trend. The main empirical facts of US business cycles are:

1. Consumption is procyclical, yet less volatile than output
2. Investment is procyclical and more volatile than output
3. Employment varies considerably and is also procyclical
4. Productivity growth (as measured by the Solow residual) is procyclical
5. Wages tend to vary with productivity and have low correlation with output

*DISCLAIMER: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even somewhat incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!




U.S. post-war Business Cycles

A complete theory of the business cycle must have two components. The first is a description of the types of shocks or disturbances believed to affect the economy the most. Examples of economic disturbances emphasized by various business cycle theories include supply shocks, changes in monetary or fiscal policy, and sudden changes in consumer spending (i.e.: demand shocks). The second component is a model that describes how key macroeconomic variables, such as output, employment, and prices, respond to economic shocks.

An influential group of classical macroeconomists, led by Nobel laureates Edward Prescott and Finn Kydland, developed a theory that takes a strong stand on the sources of shocks that cause cyclical fluctuations. This theory, the Real Business Cycle (RBC) theory, argues that real shocks to the economy are the primary cause of business cycles. Real shocks are disturbances to the "real side" of the economy, such as shocks that affect the production function, the size of the labor force, the real quantity of government purchases, and the spending and saving decisions of consumers. Economists contrast real shocks with nominal shocks, or shocks to money supply or money demand.

Although many types of real shocks could contribute to the business cycle, RBC economists assign the largest role to productivity shocks. These include the development of new products or production methods, the introduction of new management techniques, changes in the quality of capital or labor, changes in the availability of raw materials or energy, unusually good or unusually bad weather, changes in government regulations affecting production, and any other factor affecting productivity. According to this view, most economic booms result from beneficial productivity shocks, and most recessions are caused by adverse productivity shocks.

So how does the RBC theory perform when compared to US data? By combining the classical (or market-clearing) version of the IS-LM model with the assumption that productivity shocks are the dominant form of economic disturbance, RBC theory remains relatively simple yet still consistent with many of the basic business cycle facts. First, under the assumption that the economy is being continuously buffeted by productivity shocks, the RBC approach predicts recurrent fluctuations in aggregate output, which actually occur. Second, the RBC theory correctly predicts that employment will move procyclically. Third, the RBC theory predicts that real wages will be higher during booms than during recessions (procyclical real wages), as also occurs. Last, RBC theory predicts that average labor productivity is procyclical; that is, output per worker is higher during booms than during recessions. This fact is consistent with the RBC economists' assumption that booms are periods of beneficial productivity shocks, which tend to raise labor productivity, whereas recessions are the results of adverse productivity shocks, which tend to reduce labor productivity.

A business cycle fact that does not seem to be consistent with the simple RBC theory is that inflation tends to slow during or immediately after a recession. The theory predicts that an adverse productivity shock will both cause a recession and increase the general price level. Thus, according to the RBC approach, periods of recession should also be periods of inflation, contrary to the business cycle fact.

# 1 The Model

Probably the best known paper in the RBC literature is Kydland and Prescott (1982). This paper introduces both a specific theory of business cycles as well as a methodology for testing competing theories. The RBC model will be based in the neoclassical growth model adding both a labor-leisure choice and a stochastic technology productivity process. This produces what's commonly known as the "baseline real business cycle model".

**Households**

Households in this economy value consumption and leisure. They can invest in a risky asset by accumulating capital and/or in a one-period risk free bond. Agents supply labor $h_t \in [0, 1]$ in period $t$ and face uncertainty about future prices so seeks to maximize lifetime expected utility:

$$\begin{aligned}
\max_{\{c_t, h_t, b_{t+1}, i_t\}_{t=0}^{\infty}} U &= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \\
s.t. \quad &: \quad w_t h_t + r_t k_t + R_t b_t + \pi_t \geq x_t + c_t + b_{t+1} \\
&: \quad k_{t+1} = (1 - \delta) k_t + i_t \\
&: \quad k_t \geq 0 \\
&: \quad k_t > 0 \text{ given} \\
&: \quad \text{NPG condition}
\end{aligned}$$

We assume that the household is making all time-t choices, conditional on time $t$ information.

**Firm**

The representative firm faces a static optimization problem and is subject to random productivity shocks. The firm's problem is:

$$\begin{aligned}
\max_{k_t, h_t} \pi_t &= p_t y_t - w_t h_t - r_t k_t \\
s.t. \quad &: \quad y_t = e^{z_t} F(k_t, h_t) \\
&: \quad z_t = \rho z_{t-1} + \varepsilon_t
\end{aligned}$$

where $\varepsilon_t$ is white noise and $F(k_t, h_t)$ is CRS.

**Solving the model:**

The agent's lagrangean is:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1-n_t) + \lambda_t [w_t h_t + r_t k_t + R_t b_t + \pi_t - k_{t+1} + (1-\delta)k_t - c_t - b_{t+1}] \right\}$$

The first order conditions are:

$$\begin{aligned}
\frac{\delta L}{\delta c_t} &= 0 \iff u_c(c_t, 1-h_t) = \lambda_t \\
\frac{\delta L}{\delta h_t} &= 0 \iff u_l(c_t, 1-h_t) = \lambda_t w_t \\
\frac{\delta L}{\delta k_{t+1}} &= 0 \iff \beta E_t[\lambda_{t+1}(r_{t+1} + 1 - \delta)] = \lambda_t \\
\frac{\delta L}{\delta b_{t+1}} &= 0 \iff \beta E_t[\lambda_{t+1} R_{t+1}] = \lambda_t
\end{aligned}$$

substitution yields the intratemporal condition:

$$u_l(c_t, 1-h_t) = u_c(c_t, 1-h_t) w_t$$

showing that the agent equates the marginal utility from a little more leisure to the utility from working an equal amount and getting more consumption. Additionally, we have:

$$\begin{aligned}
u_c(c_t, 1-h_t) &= \beta E_t[u_c(c_{t+1}, 1-h_{t+1})(r_{t+1} + 1 - \delta)] \\
&= \beta[E_t(u_c(c_{t+1}, 1-h_{t+1}))(E_t(r_{t+1}) + 1 - \delta) + Cov(u_c(c_{t+1}, 1-h_{t+1}), r_{t+1})]
\end{aligned}$$

which is the intratemporal condition also known as Euler equation. Note that because the agents face uncertainty about the return on capital, today's investment decision doesn't just depend on the expected value of future prices, but also on their covariance with consumption [1].

The solution to the firm's problem yields:

$$\begin{aligned}
\frac{\delta \pi_t}{\delta h_t} &= 0 \iff e^{z_t} F_H(k_t, h_t) = w_t \\
\frac{\delta \pi_t}{\delta k_t} &= 0 \iff e^{z_t} F_K(k_t, h_t) = r_t
\end{aligned}$$

[1] Remember that for 2 random variables $X$ and $Y$: $E(XY) = E(x)E(Y) + Cov(X, Y)$

**Calibration:**

Kydland and Prescott suggest a way to identify if this model can explain business cycles. The basic idea is to:

- Choose the model's parameters ($\beta$, $\delta$, etc.) based on evidence outside of the model (e.g.: microeconomic studies or theory)
- Solve the model numerically and simulate the economy
- Compare the moments (standard deviations, correlations, etc) of the simulated economy with moments from actual data
- If the moments are matched, we are done. If not, there might be room for improvement.

To do this, we would first need to put some parametric functional forms into the model above. First, utility:

$$u(c_t, 1-h_t) = \frac{\left(c_t^{1-\alpha}(1-h_t)^{\alpha}\right)^{1-\gamma}}{1-\gamma}$$

which is a CRRA utility function, with intertemporal elasticity of substitution equal to $\frac{1}{\gamma}$. Production is assumed to be Cobb-Douglas:

$$F(k_t, h_t) = k_t^{\theta} h_t^{1-\theta}$$

Now we need plausible parameter values for this model. I'll use the values from Cooley and Prescott (1995), which is an identical model expect for the technology and population trends.

For $\beta$ we have that at the non-stochastic steady state, $R = \frac{1}{\beta}$. The average real interest rate in the US is usually around 4% annually, which is about 1% per quarter. Since we are working with quarterly data and $R_t$ is the gross interest rate, we have that $R - 1 = \frac{1}{\beta} - 1 = 0.01$ or $\beta = 0.99$.

For $\theta$ we know that $1 - \theta$ will be labour's share of output, a quantity that can be estimated from the national income accounts. Cooley and Prescott set it to 0.40.

For $\gamma$ we employ estimates from micro studies which place the typical worker's intertemporal elasticity of substitution in a range around 1. With $\gamma = 1$ we obtain the following utility function:

$$u(c_t, h_t) = (1-\alpha)\ln c_t + \alpha \ln(1-h_t)$$

Now from the intratemporal condition and assuming we are at steady state:

$$\frac{\alpha}{1-\bar{h}} = \frac{1-\alpha}{\bar{c}} w$$

substitute in the FOC for the firm yields:

$$\frac{\alpha}{1-\bar{h}} = \frac{1-\alpha}{\bar{c}}(1-\theta)\frac{\bar{y}}{\bar{h}}$$

We know fix $\bar{h} = 0.31$ representing that in the non-stochastic steady state agents work about 31 % of their time endowment. Moreover, we also fix the steady state output to consumption ratio about 1.33 (implies that $\frac{c}{y} = 0.75$). Substituting into the equations above, we obtain $\alpha = 0.64$.

In terms of the depreciation parameter, Cooley and Prescott estimate that US depreciation is about 4.8 % annually, or 1.2 % quarterly. Hence, they set $\delta = 0.012$. Last, the parameters for the stochastic process for technology need to be estimated. Since this model has perfect competition and constant returns to scale in production, $z_t - z_{t-1}$ can be estimated as the average value for the Solow residual. Colley and Prescott set average productivity growth rate (the technology trend) at 0.0156 or 1.6% annual TFP growth. Once we subtract this average from the data, we can estimate an AR(1) model and obtain $\rho = 0.95$ and $\sigma = 0.007$.

Last, it should be mentioned that this is not the only approach to compare a model with the data. Alternative approaches are:

- General Method of Moments
- Maximum likelihood estimation
- Bayesian estimation

In these cases we are choosing the parameters so that the model produces data that resembles the actual data as much as possible. An important underlying assumption made, is that the model is the correct one and we are just trying to pin down the parameters values. If some of the parameter values obtain seemed unreasonable, this could point towards some issues with the model.

**Solution and Simulation:**

Once all parameter values have been calibrated/estimated, we need to find a solution to the model. In few very rare cases we'll be able to find a close form solution[2] and so more often than not we shall rely on numerical methods[3]. Common techniques are value function iteration, or approximations of the policy functions around the steady state. This note will not cover any method in detail, although it will provide an example of how to log-linearize an RBC model.

Once the model has been solved, we can simulate a series of stochastic disturbances on a computer and generate simulated series for output, employment, consumption, etc. We can then use this simulated data to compare their properties to the properties of the corresponding detrended variables from the economy under study. The table below provides an example:

| | US data | | Baseline RBC model | |
|---|---|---|---|---|
| | Std. Dev (%) | $\rho_{x,GNP}$ | Std. Dev (%) | $\rho_{x,GNP}$ |
| GNP | 1.72 | 1.0 | 1.35 | 1.0 |
| Consumption (NDS) | 0.86 | 0.77 | 0.329 | 0.843 |
| Investment (GPDI) | 8.24 | 0.91 | 5.954 | 0.992 |
| Hours | 1.59 | 0.86 | 0.769 | 0.986 |
| Productivity | - | - | 0.606 | 0.978 |

As in reality consumption is much less variable than output, while investment is more volatile. Similarly, both variables strongly correlate with GDP. There are a few inconsistencies though. The model seems to significantly understate the variability of both consumption and hours, and less so that of output. Overall, however, it is a pretty good fit.

[2]See notes on solving DSGE models by hand.
[3]See notes on solving DSGE models.

## 2 RBC Linearized Solution

This section provides an example on how to linearize an RBC model around its steady state. The model is a slight variation of Hansen (1985) real business cycle model.

**The set up:**

$$\text{[CRRA utility]} \quad : \quad U = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\eta} - 1}{1-\eta} - AN_t \right] \tag{1}$$

$$\text{[Resource constraint]} \quad : \quad Y_t \geq C_t + I_t \tag{2}$$

$$\text{[LOM for Investment]} \quad : \quad K_t = I_t + (1-\delta)K_{t-1} \tag{3}$$

$$\text{[Production technology]} \quad : \quad Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha} \tag{4}$$

$$\text{[LOM of shock]} \quad : \quad \ln(Z_t) = (1-\psi)\ln(\bar{Z}) + \psi \ln(Z_{t-1}) + \varepsilon_t : \varepsilon_t \sim N(0, \sigma^2) \tag{5}$$

$$\text{[Initial conditions]} \quad : \quad Z_0 \vee K_0 \tag{6}$$

where $\beta, \eta, \psi, \sigma^2, \alpha$, $A$ and $\bar{Z}$ are all known parameters.

In particular, Hansen considered the case where $\eta \to 1$, so that the objective function becomes:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - AN_t]$$

**The problem:**

The social planner seeks to maximize the representative's agent lifetime utility (1) subject to (2) - (5), given (6). We can think of the agent's optimization problem in the following sequential way: given a predetermined capital stock level $K_t$ and a particular shock realization, output levels are revealed. Once agents are aware of how much production is available they optimize to determine how much to consume and how much to invest.

We can somewhat simplify the problem by:

$$\text{Combining (2) and (3) yields} \quad : \quad Y_t \geqslant C_t + K_{t+1} - (1-\delta)K_t \tag{7}$$

$$\text{Combining (4) and (7) yields} \quad : \quad Z_t K_t^{\alpha} N_t^{1-\alpha} \geqslant C_t + K_{t+1} - (1-\delta)K_t \tag{8}$$

**The lagrangian:**

$$\max_{\{C_t,N_t,K_{t+1}\}_{t=0}^{\infty}} U = \max_{\{C_t,N_t,K_{t+1}\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - AN_t + \lambda_t (Z_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1-\delta)K_t)] \right\} \tag{9}$$

**The FOCs:**

$$[C_t] \quad : \quad C_t^{-1} = \lambda_t \tag{10}$$

$$[N_t] \quad : \quad A = \lambda_t (1-\alpha) Z_t K_t^{\alpha} N_t^{-\alpha} \tag{11}$$

$$[K_{t+1}] \quad : \quad -\lambda_t + \beta E_t \lambda_{t+1} [Z_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta)] = 0 \tag{12}$$

$$[\lambda_t] \quad : \quad Z_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1-\delta)K_t = 0 \tag{13}$$

Note that:

(i) The need for the term $E_t$ (expectation conditional on the information set available at time $t$) arises from the fact that at the time when $K_t$ is decided, agents ignore the realization of the shock in "$t+1$" as well as the value of the lagrange multiplier in the same period. As such, the relevant values of $Z_{t+1}$ and $\lambda_{t+1}$ will be their expected values at time "$t$".

(ii) Given that agents do value leisure, the social planner will take this into consideration and seek to equate the marginal utility of consumption times the marginal product of labour to the marginal desutility of labour (or the marginal utility of leisure).

Combining (10) and (12) yields the Euler equation for consumption:

$$\begin{aligned} \beta E_t C_{t+1}^{-\eta} [Z_{t+1} \alpha K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta)] &= C_t^{-1} \\ &\Rightarrow 1 = \beta E_t \left( \frac{C_t}{C_{t+1}} \right) [\alpha Z_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta)] \end{aligned} \tag{14}$$

If we define the Gross Return on capital as: $R_t = \alpha Z_t K_t^{\alpha-1} N_t^{1-\alpha} + (1-\delta)$, we can re-write (14) as:

$$1 = \beta E_t \left( \frac{C_t}{C_{t+1}} \right) R_{t+1} \tag{15}$$

**Solving for the steady states:**

So far we have the following equations characterizing the equilibrium path:

A resource constraint that needs to be satisfied every period: $Z_t K_t^{\alpha} N_t^{1-\alpha} - K_t + (1-\delta)K_{t-1} = C_t$

An intratemporal optimality condition that needs to be satisfied every period: $A = \lambda_t (1-\alpha) Z_t K_t^{\alpha} N_t^{-\alpha}$

An intertemporal optimality condition that needs to be satisfied every period: $1 = \beta E_t \left(\frac{C_t}{C_{t+1}}\right) R_{t+1}$

A set of initial conditions: $Z_0 \vee K_0 > 0$

A terminal condition: $\lim_{t\to\infty} E_0[\beta^t C_t^{-1} K_t] = 0$

A LOM of the shock: $\ln(Z_t) = (1-\psi)\ln(\bar{Z}) + \psi \ln(Z_{t-1}) + \varepsilon_t$

Note, the return equation can also be written as: $R_t = \alpha \frac{Y_t}{K_t} + (1-\delta)$

Dropping time indices yields:

$$\bar{C} = \bar{Z}\bar{K}^{\alpha} - \bar{K} + (1-\delta)\bar{K} \tag{16}$$

$$1 = \beta \bar{R} \tag{17}$$

$$\bar{R} = \alpha \frac{\bar{Y}}{\bar{K}} + (1-\delta) \tag{18}$$

The idea is to write every variable in terms of known parameters. Hence:

From [17] $\bar{R} = \frac{1}{\beta}$

From [18] $\bar{R} - (1-\delta) = \alpha \bar{Z}\bar{K}^{\alpha-1} \Rightarrow \bar{K} = \left(\frac{\bar{R}-(1-\delta)}{\alpha \bar{Z}\bar{N}^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$

Knowing $\bar{K}$ makes it obtaining $\bar{C}$ straight forward.

**Performing a linear approximation around the steady state:**

Knowing that: $X_t \simeq \bar{X}(1+\hat{x}_t) \vee X_t^{\alpha} \simeq \bar{X}(1+\alpha\hat{x}_t)$ the next step is to log-linearize each of the equations that characterize the equilibrium around the steady state:

From the resource constraint: $Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} - K_t + (1-\delta)K_{t-1} = C_t$

$\bar{C}(1+\hat{c}_t) \simeq \bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}(1+\hat{z}_t)(1+\alpha\hat{k}_{t-1})(1+(1-\alpha)\hat{n}_t) - \bar{K}(1+\hat{k}_t) + (1-\delta)\bar{K}(1+\hat{k}_{t-1})$

$\bar{C}(1+\hat{c}_t) \simeq \bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}(1+(1-\alpha)\hat{n}_t + \alpha(1-\alpha)\hat{n}_t\hat{k}_{t-1} + \hat{z}_t + \alpha\hat{k}_{t-1} + \alpha\hat{z}_t\hat{k}_{t-1} + (1-\alpha)\hat{n}_t\hat{z}_t + \alpha(1-\alpha)\hat{n}_t\hat{z}_t\hat{k}_{t-1}) - \bar{K}(1+\hat{k}_t) + (1-\delta)\bar{K}(1+\hat{k}_{t-1})$

where $\hat{z}_t\hat{k}_{t-1}, \hat{n}_t\hat{z}_t\hat{k}_{t-1}, \hat{n}_t\hat{z}_t$, are assumed to be very small and hence $= 0$ :

$$\Rightarrow \bar{C}(1+\hat{c}_t) \simeq \bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}(1+\hat{z}_t+\alpha\hat{k}_{t-1}+(1-\alpha)\hat{n}_t) - \bar{K}(1+\hat{k}_t) + (1-\delta)\bar{K}(1+\hat{k}_{t-1}) \tag{19}$$

Next, (19) can be re written using the steady state relationships:

$\bar{Y} = \bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}$

$\bar{C} = \bar{Y} - \delta\bar{K}$

$\bar{R} = \alpha\bar{Z}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + (1-\delta) = \alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)$

$A = (1-\alpha)\bar{Z}\bar{K}^{\alpha}\bar{N}^{-\alpha}\bar{\lambda}$

$\Rightarrow \bar{Y} - \delta\bar{K} + \bar{C}\hat{c}_t \simeq \bar{Y}(1+(1-\alpha)\hat{n}_t+\hat{z}_t+\alpha\hat{k}_{t-1}) - \bar{K}(1+\hat{k}_t) + (1+\hat{k}_{t-1}-\delta-\delta\hat{k}_{t-1})\bar{K}$

$\Rightarrow 1 - \delta\frac{\bar{K}}{\bar{Y}} + \frac{\bar{C}}{\bar{Y}}\hat{c}_t \simeq 1+(1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t - \frac{\bar{K}}{\bar{Y}}(1+\hat{k}_t) + (1+\hat{k}_{t-1}-\delta-\delta\hat{k}_{t-1})\frac{\bar{K}}{\bar{Y}}$

$\Rightarrow \frac{\bar{C}}{\bar{Y}}\hat{c}_t \simeq (1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t + \frac{\bar{K}}{\bar{Y}}(-1-\hat{k}_t+\delta+1+\hat{k}_{t-1}-\delta-\delta\hat{k}_{t-1})$

$\Rightarrow \frac{\bar{C}}{\bar{Y}}\hat{c}_t \simeq (1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t + \frac{\bar{K}}{\bar{Y}}(-\hat{k}_t+\hat{k}_{t-1}-\delta\hat{k}_{t-1})$

$\Rightarrow \frac{\bar{C}}{\bar{Y}}\hat{c}_t - \frac{\bar{K}}{\bar{Y}}(-\hat{k}_t+\hat{k}_{t-1}-\delta\hat{k}_{t-1}) \simeq (1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t$

Given that:

$\bar{C}\hat{c}_t + \bar{I}\hat{\imath}_t = \bar{Y}\hat{y}_t \Rightarrow \hat{y}_t = \frac{\bar{C}}{\bar{Y}}\hat{c}_t + \frac{\bar{I}}{\bar{Y}}\hat{c}_t \wedge \bar{I}\hat{\imath}_t = \bar{K}\hat{k}_t + (1-\delta)\bar{K}\hat{k}_{t-1}$

We get that:

$$\Rightarrow \hat{y}_t \simeq (1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t \tag{20}$$

For the return equation: $R_t = \alpha\frac{Y_t}{K_{t-1}} + (1-\delta)$

Given $Y_t = e^{\ln(y_t)} = \bar{Y}e^{\hat{y}_t}$ we can rewrite the above as:

$\bar{R}(1+\hat{r}_t) = \alpha\frac{\bar{Y}e^{\hat{y}_t}}{\bar{K}e^{\hat{k}_{t-1}}} + (1-\delta)$

Since $\bar{R} = \alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)$ we have that:

$\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta) + \bar{R}\hat{r}_t = \alpha\frac{\bar{Y}e^{\hat{y}_t}}{\bar{K}e^{\hat{k}_{t-1}}} + (1-\delta)$

$\Rightarrow \alpha\frac{\bar{Y}}{\bar{K}} + \bar{R}\hat{r}_t = \alpha\frac{\bar{Y}e^{\hat{y}_t}}{\bar{K}e^{\hat{k}_{t-1}}}$

$\Rightarrow \bar{R}\hat{r}_t = \alpha\frac{\bar{Y}}{\bar{K}}(e^{\hat{y}_t-\hat{k}_{t-1}}-1)$

Since $e^x \simeq (1+x)$ then:

$\Rightarrow \bar{R}\hat{r}_t \simeq \alpha\frac{\bar{Y}}{\bar{K}}(1+\hat{y}_t-\hat{k}_{t-1}-1)$

$$\Rightarrow \bar{R}\hat{r}_t \simeq \alpha\frac{\bar{Y}}{\bar{K}}(\hat{y}_t-\hat{k}_{t-1}) \tag{21}$$

For the optimality condition: $A = \lambda_t(1-\alpha)Z_tK_{t-1}^{\alpha}N_t^{-\alpha}$

$A = (1-\alpha)\bar{Z}\bar{K}^{\alpha}\bar{N}^{-\alpha}\bar{\lambda}(1+\hat{z}_t)(1+\alpha\hat{k}_{t-1})(1-\alpha\hat{n}_t)(1+\hat{\lambda}_t) = (1-\alpha)\bar{Z}\bar{K}^{\alpha}\bar{N}^{-\alpha}\bar{\lambda}(1+\hat{z}_t+\alpha\hat{k}_{t-1}-\alpha\hat{n}_t+\hat{\lambda}_t)$

$\Rightarrow A = A(1+\hat{z}_t+\alpha\hat{k}_{t-1}-\alpha\hat{n}_t+\hat{\lambda}_t)$

$\Rightarrow 1 = 1+\hat{z}_t+\alpha\hat{k}_{t-1}-\alpha\hat{n}_t+\hat{\lambda}_t$

$\Rightarrow 0 = \hat{z}_t+\alpha\hat{k}_{t-1}-\alpha\hat{n}_t+\hat{\lambda}_t$

Adding and subtracting $\hat{n}_t$ yields:

$\Rightarrow 0 = \hat{z}_t+\alpha\hat{k}_{t-1}-\alpha\hat{n}_t+\hat{\lambda}_t+\hat{n}_t-\hat{n}_t$

Given that $\hat{y}_t \simeq (1-\alpha)\hat{n}_t+\alpha\hat{k}_{t-1}+\hat{z}_t$ :

$0 \simeq \hat{y}_t-\hat{n}_t+\hat{\lambda}_t$

$\Rightarrow 0 \simeq \hat{y}_t-\hat{n}_t-\hat{c}_t$

$$\Rightarrow \hat{y}_t \simeq \hat{n}_t+\hat{c}_t \tag{22}$$

For the Euler equation: $1 = \beta E_t\left(\frac{C_t}{C_{t+1}}\right)^{\eta}R_{t+1}$

From log-linearizing we know that $X_t \simeq \bar{X}(1+\hat{x}_t) \vee e^{\hat{x}_t} \simeq (1+\hat{x}_t) \Rightarrow X_t \simeq \bar{X}e^{\hat{x}_t}$

With this in mind, we can re-write $C_t \simeq \bar{C}e^{\hat{c}_t} \vee C_{t+1} \simeq \bar{C}e^{\hat{c}_{t+1}}$

and then re-write the Euler equation as: $1 \simeq E_t[\beta\left(e^{\hat{c}_t-\hat{c}_{t+1}}\right)^{\eta}\bar{R}e^{\hat{r}_{t+1}}]$

Again using the substitution $e^{\hat{x}_t} \simeq (1+\hat{x}_t)$ :

$1 \simeq E_t[(1+\eta(\hat{c}_t-\hat{c}_{t+1}))(1+\hat{r}_{t+1})]$

$0 \simeq E_t[\eta(\hat{c}_t-\hat{c}_{t+1})+\hat{r}_{t+1}+\eta(\hat{c}_t-\hat{c}_{t+1})\hat{r}_{t+1}]$

Given that $\eta(\hat{c}_t-\hat{c}_{t+1})\hat{r}_{t+1}$ is not linear and considered second order, it is assumed to be equal to zero. Hence:

$$0 \simeq E_t[\eta(\hat{c}_t-\hat{c}_{t+1})+\hat{r}_{t+1}] \tag{23}$$

For the shock equation:

$\ln(\bar{Z}_t) = (1-\psi)\ln(\bar{Z})+\psi\ln(Z_{t-1})+\varepsilon_t$

$\ln(\bar{Z}e^{\hat{z}_t}) = (1-\psi)\ln(\bar{Z})+\psi\ln(\bar{Z}e^{\hat{z}_{t-1}})+\varepsilon_t$

$$\Rightarrow \hat{z}_t = \psi \hat{z}_{t-1} + \varepsilon_t \tag{24}$$

All in all we end up with a system of six linear equations:

$$\begin{aligned}
\hat{y}_t &\simeq (1-\alpha)\hat{n}_t + \alpha \hat{k}_{t-1} + \hat{z}_t && (25)\\
\bar{R}\hat{r}_t &\simeq \alpha \frac{\bar{Y}}{\bar{K}}(\hat{y}_t - \hat{k}_{t-1}) && (26)\\
\hat{y}_t &\simeq \hat{n}_t + \hat{c}_t && (27)\\
\hat{z}_t &= \psi \hat{z}_{t-1} + \varepsilon_t && (28)\\
0 &\simeq E_t[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}] && (29)\\
\hat{y}_t &\simeq \hat{c}_t + \hat{k}_t - (1-\delta)\hat{k}_{t-1} && (30)
\end{aligned}$$

and six unknowns:

$$\hat{y}_t, \hat{n}_t, \hat{k}_t, \hat{z}_t, \hat{c}_t, \hat{r}_t$$

Moreover, these unknowns can be further classified as state (both exogenous and endogenous) and control variables. The endogenous state variable is $k_t$, while the exogenous one is $z_t$. The controls are $c_t, r_t, n_t, y_t$. One may then use numerical algorithms or matrix manipulation methods to solve the above system of linear equations.

## 3 Additional Practice Problem

Suppose there is a decentralized economy in which agents maximize preferences given by:

$$\begin{aligned} \max_{\{c_t,h_t,k_{t+1}\}} U &= E_0 \sum_{t=0}^{\infty} \beta^t [\ln c_t + \theta \ln(1-h_t)] \\ s.t. &: w_t h_t + r_t k_t \geq c_t + k_{t+1} - (1-\delta)k_t \\ &: 0 < \beta < 1, 0 < \alpha < 1, 0 < \delta < 1, k_0 > 0 \text{ given} \end{aligned}$$

and the representative firm solves:

$$\begin{aligned} \max_{(h_t,k_t)} \pi_t &= y_t - w_t h_t - r_t k_t \\ s.t. &: y_t = e^{z_t} k_t^{\alpha} h_t^{1-\alpha} \\ &: z_t = \rho z_{t-1} + \epsilon_t \\ &: \epsilon_t \sim N(0,\sigma^2), \rho \in [0,1) \end{aligned}$$

**Questions:**

1. Define there first order conditions for $c_t, h_t$ and $k_{t+1}$.
2. Derive the equilibrium conditions (you should get 7 equations, 7 endogenous variables and 1 exogenous variable).

**Answers:**

(1) The Household's lagrangean is:

$$L = \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \theta \ln(1-h_t) + \lambda_t [w_t h_t + r_t k_t - c_t - k_{t+1} + (1-\delta)k_t]\}$$

The first order conditions are:

$$\begin{aligned} \frac{\delta L}{\delta c_t} = 0 &\iff \frac{1}{c_t} = \lambda_t \\ \frac{\delta L}{\delta h_t} = 0 &\iff \lambda_t w_t = \frac{\theta}{1-h_t} \\ \frac{\delta L}{\delta k_{t+1}} = 0 &\iff \lambda_t = \beta E_t[\lambda_{t+1}(r_{t+1} + 1 - \delta)] \end{aligned}$$

The Firm's first order conditions are:

$$
\begin{aligned}
\frac{\delta \pi_t}{\delta k_t} = 0 \quad &\Longleftrightarrow \quad r_t = \alpha e^{z_t} k_t^{\alpha-1} h_t^{1-\alpha} \\
\frac{\delta \pi_t}{\delta h_t} = 0 \quad &\Longleftrightarrow \quad w_t = (1-\alpha) e^{z_t} k_t^{\alpha} h_t^{-\alpha}
\end{aligned}
$$

(2) The 7 endogenous variables are: $c_t, h_t, k_{t+1}, w_t, r_t y_t$ and $i_t$. The exogenous one is $z_t$.

The 7 equilibrium conditions are:

$$
\begin{aligned}
w_t &= \frac{\theta c_t}{1-h_t} \\
1 = &= \beta \left( E_t \frac{c_t}{c_{t+1}} \right) [r_{t+1} + 1 - \delta] \\
y_t &= e^{z_t} k_t^{\alpha} h_t^{1-\alpha} \\
i_t &= k_{t+1} - (1-\delta) k_t \\
r_t &= \alpha \frac{y_t}{k_t} \\
w_t &= (1-\alpha) \frac{y_t}{h_t} \\
y_t &= c_t + i_t
\end{aligned}
$$

## Developing some intuition:

Consider an RBC model with standard log preferences like the one described above. We seek to understand the effects of TFP disturbances both with and without persistence (i.e.: an iid shock).

Assume $\{z_t\}$ is i.i.d normal. A positive TFP shock implies that both $MPK_t$ and $MPN_t$ increase:

$$
\uparrow r_t \Rightarrow \begin{cases} \uparrow c_t & \text{(income effect)} \\ \downarrow h_t & \text{(income effect)} \end{cases}
$$

Since the capital stock at time $t$ is fixed, the increase in productivity raises the return on capital $r_t$. This induces a purely income effect as the household's period income increases (i.e.: there is no relative price change). Given the agent's nomothetic preferences he would want to consume more of both normal goods (consumption and leisure in our set-up).

$$\uparrow w_t \Rightarrow \begin{cases} \uparrow c_t & \text{(income effect)} \\ \uparrow h_t & \text{(substitution effect)} \end{cases}$$

However, the productivity shock has also raised the opportunity cost of leisure as real wages have gone up due to labor's higher productivity. Again, the income effect of perceiving higher wages has the agent wanting to consume more, but the higher opportunity cost of leisure has him wanting to rest less and increase labor supply. In other words, the change in $w_t$ represents a change in relative prices between work and leisure and (usually) the substitution effect will dominate. This reaction is in fact consistent with the data, where we observe that the labor market is highly pro-cyclical.

Assume now that $\{z_t\}$ follows a process with persistence. For example:

$$\begin{aligned} z_t &= \rho z_{t-1} + \epsilon_t \\ &: \quad \epsilon_t \sim N(0, \sigma^2); \;\; \rho \in [0, 1) \end{aligned}$$

How do our results vary from before? The overall effects are similar with a few differences:
First since the agent is forward looking and shocks now have some persistence, the agent understands that if productivity gets high it will probably remain high for some time. In turn, he will seek to increase labor supply more than in the iid case, because he also knows that when productivity is low, it most likely be low for a while. As such the direction of the response remains the same, yet the magnitude will likely differ relative to the iid case.
Second, also given the shock's persistence, a higher MPK today will most likely translate into a higher future MPK. This implies a higher $r_{t+1}$ and with that a relative price change between today and tomorrow which we did not have in the iid case. As with any other relative price movement we'll have income and substitution effects considerations. The higher future rental rate of capital has the agent wanting to consume more. However, the increase in the expected $r_{t+1}$ also means that opportunity cost of consumption has gone up, having the household wanting to increase savings. Overall, under normal parametrizations, consumption tends to increase more than in the iid case.