Notes on Search Models

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In all the models we have considered so far, the market itself had a very simple structure: agents observe prices and these prices are determined anonymously by the forces of demand and supply. Search models are different. In a typical search framework an individual has only access to a few transactions at every point in time and must decide whether to accept the available offer in hand, or delay in hopes of receiving something better in the future. These models are commonly used to model unemployment, but also find applications in explaining money demand, coordination failures, etc.

A basic model

Consider a worker who seeks to maximize his lifetime discounted income:

$$\sum_{t=0}^{\infty} \beta^t y_t$$

The worker can be in one of two states: employed or unemployed, and begins as being unemployed. In each period of unemployment, the worker will be matched with a single employer who will offer wage w_t. We abstract from the wage variation process and assume that the wage offer is an independent draw from a probability distribution with cdf F(.), where two conditions hold:

$$\begin{aligned} F(0) &= 0 \\ F(w) &= 1 : \text{for some w} < \infty \end{aligned}$$

Suppose that the worker receives a wage offer of w_t in period T. The worker has two options: either accept or reject the offer. If the worker rejects the offer, he or she is unemployed and receives nothing $y_t = 0$. If the workers accepts a wage offer, he or she continues to work at that wage forever: $y_t = w_t$ for all $t \geq T$.

We can describe the worker's problem by a Bellman equation. Let $V(w_0)$ be the value function of an unemployed worker who has received a wage offer w_0. If the offer is accepted

*DISCLAIMER: I wrote these notes as a study aid for myself. They are work in progress and could be incomplete, inaccurate and even somewhat incorrect. Keep that in mind should you decide to use them. Comments and suggestions welcomed!

then:

$$V(w_0) = \sum_{t=0}^{\infty} \beta^t w_t = \frac{w_0}{1-\beta}$$

if, however, the offer is rejected we then have that:

$$V(w_0) = \beta E_0 V(w_1)$$

Putting both outcomes together yields:

$$V(w_t) = \max \left\{ \frac{w_t}{1-\beta}, \beta E_0 V(w_{t+1}) \right\}$$

The solution to this problem will in the form of a reservation wage strategy. That is to say, there is some wage offer θ which will render the worker indifferent between accepting or rejecting. The worker will accept anything $\geq \theta$ and reject anything below it. Given this logic, we hypothesis the solution would take the form:

$$V(w_t) = \begin{cases} \frac{\theta}{1-\beta} & \text{if } w_t < \theta \\ \frac{w_t}{1-\beta} & \text{if } w_t \geq \theta. \end{cases} \pmod 2$$

Therefore θ must solve:

$$V(\theta) = \frac{\theta}{1-\beta} = \beta \left[F(\theta) \frac{\theta}{1-\beta} + \int_{\theta}^{W} \frac{w_{t+1}}{1-\beta} dF(w_{t+1}) \right]$$

once we have a specific functional form for $F(.)$ solving the above becomes easy. Consider the example below.

Example

Suppose that the distribution of wage offers is uniform across the $[0, W]$ interval.

$$F(w) = \begin{cases} 0 & \text{if } w_t < 0 \\ w/W & \text{if } 0 < w < W \\ 1 & \text{if } w > W \end{cases}$$

in this case:

$$\begin{aligned} \frac{\theta}{1-\beta} &= \beta \left[\frac{\theta}{W} \frac{\theta}{1-\beta} + \int_{\theta}^{W} \frac{w}{W(1-\beta)} dw \right] \\ &= \frac{\beta}{1-\beta} \left[\frac{\theta^2}{W} + \frac{W^2 - \theta^2}{2W} \right] \end{aligned}$$

which can be solved using the methods for a quadratic equation to yield:

$$\theta = \frac{1 - \sqrt{1-\beta^2}}{\beta} W$$

where we can see that the reservation wage increases proportionally to a constant term. In particular, the reservation wage is increasing in β, implying that the more patient the worker is (i.e.: the higher the β) the higher his reservation wage will be.

A more complete framework

Consider now an alternative model where we'll be able to capture the behavior of both firms and workers. In particular, this framework will be useful to generate features that models with Walrasian markets are incapable of reproducing, such us unemployment in steady state. I begin describing a deterministic set-up and later proceed to a stochastic one. The description of the model is as follows.

Assume there are L workers (normalize to 1) that can either be unemployed (u) or employed ($1-u$). Firms, on the other hand, will be able to produce as long as they have 1 employee, and will post vacancies v whenever they don't have one. The outcome of this search process will be given by the following matching function:

$$m(u, v) = \chi u^{\xi} v^{1-\xi}$$

like in a production set-up, we assume that the matching function exhibits constant returns to scale: $m(2u, 2v) = 2m(u, v)$.

Define $\theta = v/u$ as the "vacancy-unemployment ratio", an indicator of tightness in the labor market. In addition we have that:

$$\begin{aligned} \text{P(job finding)} &= \frac{m(u,v)}{u} = \frac{\chi u^{\xi} v^{1-\xi}}{u} = \chi \left(\frac{v}{u}\right)^{1-\varepsilon} = \chi \theta^{1-\varepsilon} \equiv f(\theta) \\ \text{P(job filling)} &= \frac{m(u,v)}{v} = \frac{\chi u^{\xi} v^{1-\xi}}{v} = \chi \left(\frac{v}{u}\right)^{\varepsilon} = \chi \theta^{-\varepsilon} \equiv q(\theta) \end{aligned}$$

where

$$\frac{f(\theta)}{\theta} = \frac{\chi \theta^{1-\varepsilon}}{\theta} = \chi \theta^{-\varepsilon} = q(\theta)$$

Unemployment at steady state

To think about unemployment in steady state, we should first think about flows into and out of unemployment. The law of motion of unemployment is:

$$u_{t+1} = u_t + (1 - u_t)s - u_t f(\theta)$$

Every period, a fraction s of employed people loose their job and join the unemployment pool. The parameter s is called the employment exit probability of sometimes referred to as the “separation rate”, since it denotes the probability that a worker and a firm in an existing match are separated. In this simple version of the model, s is constant and so separations are exogenous. Additionally, every period a fraction $f(\theta_t)$ of unemployed people find a job, so they exit the unemployment pool. Note that job-to-job transitions are not considered.

We can now focus on the steady state where $u_{t+1} = u_t = \bar{u}$. In turn:

$$\begin{aligned}
\bar{u} &= \bar{u} + (1 - \bar{u})s - \bar{u} f(\theta) \\
&\Rightarrow \quad \bar{u} = \frac{s}{s + f(\bar{\theta})} \\
&= \frac{s}{s + \chi\theta^{1-\varepsilon}} \\
&= \frac{s}{s + \chi\left(\frac{v}{u}\right)^{1-\varepsilon}}
\end{aligned}$$

which represents the Beveridge curve, the downward sloping relation in the u, v plane. After some algebra, we can rearrange the above as:

$$v = \left[\frac{s}{\chi}\frac{1}{u^{\varepsilon}}(1 - \bar{u})\right]^{\frac{1}{1-\varepsilon}}$$

Below I describe each market participant’s optimization problem.

The Workers

Every period a worker faces two options: he can either engage in production (work) or he can search for a job (search). If working he receives a wage w, and if searching he receives b (unemployment benefits). The assumption is that $w > b$, so that the worker prefers working to being unemployed.

A worker can then become unemployed with probability s (exogenous and constant separation rate) and an unemployed finds a job with probability $f(\theta)$. The result of this matching has effects the next period (i.e.: in an unemployed worker finds a match at time t,, he begins work at $t + 1$.)

The traditional assumptions in terms on the utility of the worker are: (1) no disutility from working or searching, (2) linear utility in consumption (i.e.: risk neutrality), (3) no saving or borrowing. This last feature will greatly simplify the agent's consumption decision in that:

$$c_t = \begin{cases} w_t & \text{if employed} \\ b & \text{if unemployed} \end{cases}$$

We can describe the worker's trade-off in terms of its Bellman equations. Let the value of working be:

$$V^E = w + \beta[(1-s)V^E + sV^U]$$

and the value of searching (i.e.: being unemployed):

$$V^U = b + \beta[f(\theta)V^E + (1-f(\theta))V^U]$$

In turn, we can derive an expression for the surplus of being employed:

$$V^E - V^U = \frac{(w-b)(1+r)}{r+s+f(\theta)} \geq 0$$

where $\beta = \frac{1}{1+r}$ in equilibrium.

The above shows that as long as $w > b$ the worker prefers to work than to search.

The Firms

A firm can employ at most one worker per period. Hence, its output will be:

$$c_t = \begin{cases} A & \text{if 1 employee} \\ 0 & \text{if no employee} \end{cases}$$

A firm with no employee can seek to hire a worker. In order to do so he must post a vacancy at cost g. Then with probability $q(\theta$ the firm hires somebody and with probability $1 - q(\theta)$ the firm does match with any employee. In this case, firms take the matching probability $q(\theta)$ as given. In turn, the value of a firm with one worker would be:

$$J^1 = (A-w) + \beta[(1-s)J^1 + sJ^0]$$

and the value of a firm with no employees (i.e.: the value of a vacancy) is:

$$J^0 = -g + \beta[q(\theta)J^1 + (1-q(\theta))J^0]$$

Given this, should a firm post a vacancy? The answer depends on the cost and benefit trade-off faced by the firm. The cost is g today, while the expected benefit is received tomorrow: $\beta[q(\theta)J^1 + (1-q(\theta))J^0]$. In turn, if $J^0 > 0$ it is profitable to post new vacancies, while if $J^0 < 0$ it is not. In equilibrium, it turns out that $J^0 = 0$. The intuition is as follows:

If $J^0 > 0$ every firm is willing to post a vacancy, because there are strictly positive benefits from doing that. When the number of vacancies v in the economy is high, the tightness indicator $\frac{v}{u}$ will be large too. Since θ is high, the probability of finding a match will be low due to the fact that many firms are looking to hire. As $q(\theta)$ falls, the probability of hiring somebody and getting revenues tomorrow declines, so the value of posting the vacancy V^0 goes down. The process continues until eventually $J^0 = 0$.

If $J^0 < 0$ the reverse occurs. With $J^0 < 0$ no firm is willing to hire because the returns of doing so will not cover the vacancy cost. If v tends to zero, θ does too, while $q(\theta)$ tends to one. In other words, an entrepreneur who posts a vacancy will find a worker almost surely. Thus entrepreneurs will actually post vacancies, driving v and θ up until $J^0 = 0$.

Given that in equilibrium $J^0 = 0$, this allows us to pin down the value of J^1. Rewritng the above expressions for J^1 and J^0:

$$\begin{aligned} J^1 &= (A-w) + \beta(1-s)J^1 \\ &= \frac{(A-w)(1+r)}{r+s} \end{aligned}$$

$$\begin{aligned} 0 &= -g + \frac{1}{1+r}q(\theta)J^1 \\ &\Rightarrow \quad J^1 = \frac{g(1+r)}{q(\theta)} \end{aligned}$$

equating both expression yields:

$$\begin{aligned} \frac{(A-w)}{r+s} &= \frac{g}{q(\theta)} \\ &\Rightarrow \quad w = A - \frac{(r+s)g}{q(\theta)} \\ &= A - \frac{(r+s)g\theta^{\varepsilon}}{\chi} \end{aligned}$$

which is sometimes referred to as the job creation equation. For a given u, if w is high few vacancies are posted and if w is low many vacancies will be posted.

The Equilibrium wage

In a frictionless neoclassical setting the equilibrium wage will equal the marginal productivity of labor. In this framework, that would be equivalent to A. However, the equilibrium wage would not be equal to A as firms would not be able to repay the vacancy cost g in that case. It turns out that in these models, both the worker and the firm will have a surplus from being in a relationship:

- If an employee walks away from its current job, he or she becomes unemployed and has to search for a new match. The loss for the worker if a separation occurs is $V^E - V^U \geq 0$
- If a firm decides to fire a worker, then it has to post a new vacancy (incurring in cost g) and search for a new worker. The loss for the firm if separation occurs is $J^1 - J^0 = J^1 > 0$

This is usually referred to as the "value of the match."

Since both the firm and the worker have surpluses from being in a match, the natural question to ask is how is this surplus split. There are several possibilities, although the most common approach is to assume that the wage is set so that both parties receive the same surplus. In turn, w solves:

$$V^E - V^U = J^1 - J^0$$

and will be given by:

$$w = \frac{b[r + s + q(\theta) + (A + g)[r + s + f(\theta)]]}{2r + 2s + q(\theta) + f(\theta)}$$

Another important results is that $\frac{\delta w}{\delta \theta} > 0$:

$$\begin{aligned}
\frac{\delta w}{\delta \theta} &= \frac{bq'(\theta) + (A+g)f'(\theta)}{2r + 2s + q(\theta) + f(\theta)} - \frac{\{b[r + s + q(\theta) + (A+g)[r + s + f(\theta)]]\}[q'(\theta) + f'(\theta)]}{[2r + 2s + q(\theta) + f(\theta)]^2} \\
&= \frac{\varepsilon \chi \theta^{-\varepsilon - 1}(w - b) + (1 + \varepsilon)\chi \theta^{-\varepsilon}(A + g - w)}{2r + 2s + q(\theta) + f(\theta)} > 0
\end{aligned}$$

The intuition is that as *theta* rises there are more vacancies relative to unemployed and so it is easier for a worker to find a job. In turn, the bargaining power of the worker goes up and the firm ends up paying a higher wage.

Equilibrium

Using the wage equation and the job creating equation, we can compute the equilibrium values of θ and w: $\hat{\theta}$ and $\hat{w}$. Given this, we can use the relationship $v = \theta u$ to find out the equilibrium values for unemployment and vacancies: $\hat{u}$ and $\hat{v}$.

Comparative Statics

What happens to the model's steady state when the economy experiences a technological improvement ($A \uparrow$)? Both the job creation and the wage equations will shift up. This implies that the equilibrium wage rises eventually leading to a rise in θ. This is similar to an RBC model where an increase in technology leads to an increase in the equilibrium number of hours worked. In the case of a search and matching model, firms want to hire more workers which pushes *theta* up, rising v and decreasing u.

The intuition is again that as A rises, it is more convenient for firms to hire workers. In order to achieve this, firms post vacancies. As more vacancies v are posted the labor market becomes crowded and it is easier for a worker to find a job. Workers are less afraid of unemployment and gain bargaining power. Firms, must offer a higher wage, reducing the convenience of hiring a worker, which partially reducing the initial increase in v and θ. However, in a calibrated version of the model θ and v rise, while u decrease.

Dynamic version of the model

I now consider a dynamic version of the same search model, where technology will evolve according to the following law of motion:

$$\ln A_t = \rho \ln A_{t-1} + \eta_t$$

For notational convenience let $f_t \equiv f(\theta_t)$ and $q_t \equiv q(\theta_t)$. Both workers and firms will now have expectations about the realization of the level of technology in the economy. The surplus for the workers now become:

$$\begin{aligned} V_t^E(A_t, u_t) &= w_t + \beta E_t[(1-s)V_{t+1}^E + sV_{t+1}^U] \\ V_t^U(A_t, u_t) &= b + \beta E_t[f_t V_{t+1}^E + (1-f_T)V_{t+1}^U] \end{aligned}$$

similarly, for firms we have that:

$$\begin{aligned} J^1 &= (A_t - w_t) + \beta E_t[(1-s)J_{t+1}^1 + sJ_{t+1}^0] \\ J^0 &= -g + \beta E_t[q_t J_{t+1}^1 + (1-q_t)J_{t+1}^0] \end{aligned}$$

The model does not have a closed form solution now and must be solved numerically in the computer. The equations characterizing the equilibrium would be:

$$\begin{aligned} f_t &= \chi\theta_t^{1-\varepsilon} \\ q_t &= \chi\theta_t^{-\varepsilon} \\ \theta_t &= \frac{v_t}{u_t} \\ u_{t+1} &= u_t + (1-u_t)s - u_t f_t \\ J_t^0 &= 0 \end{aligned}$$

Using a generalized Nash wage bargaining rule $(V_t^E - V_t^U)(1-\phi) = \phi J_t^1$, and calibrating $\phi = 0.72$ (Shimmer 2005) one can simulate the model. As in the static case, changes in technology will lead to an increase in θ_t and v_t, and a decrease in u_t.